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AUTOCORRELATED DIFFERENCES IN PAIRED SAMPLES: THE ASYMPTOTIC DISTRIBUTION OF THE SQUARE OF THE T STATISTIC FOR A FIRST ORDER AUTOREGRESSIVE GAUSSIAN PROCESS

by

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research, for acceptance, a thesis entitled AUTOCORRELATED DIFFERENCES IN PAIRED SAMPLES: THE ASYMPTOTIC DISTRIBUTION OF THE SQUARE OF THE T STATISTIC FOR A FIRST ORDER AUTOREGRESSIVE GAUSSIAN PROCESS, submitted by Jeffrey Craig Babb in partial fulfillment of the requirements for the degree of Master of Science.

ABSTRACT

The purpose of this thesis is to examine the distribution of the square of the T statistic, under the assumption that the data are sampled from a stationary first order autoregressive Gaussian process. For a sample consisting of two observations the exact probability density of $F = T^2$ is derived, while for large samples, asymptotic approximations to the probability density and moments of F are obtained.

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CHAPTER I

INTRODUCTION

Consider two times series

$$X(t) = \mu(t) + \varepsilon(t)$$

and

$$Y(t) = \nu(t) + \eta(t)$$

for all times $t \in T$, where $\varepsilon(t)$ and $\eta(t)$ are random error terms. Let $t_1 < t_2 < \ldots < t_n$ be a finite set of time points and let $(x_1,y_1) = (x(t_1),y(t_1)), (x_2,y_2) = (x(t_2),y(t_2)),\ldots,(x_n,y_n) = (x(t_n),y(t_n))$ be n pairs of observations from the two time series. We wish to test the null hypothesis

$$H_0: \mu(t) = \nu(t) \quad , \quad t \in T$$

against the alternative

$$H_A: \mu(t) \neq \nu(t) \ , \quad t \in T$$

on the basis of the n differences

$$d_i = x_i - y_i \ , \quad i = 1,2,\ldots,n \ .$$

If H_0 is true then

$$d_i = \varepsilon(t_i) - \eta(t_i) = \gamma_i \ , \quad i = 1,2,\ldots,n \ .$$

If, furthermore, the $\{\gamma_i\}$ are independent $N(0,1)$ random variables then the statistic

$$T = \frac{\bar{d}}{S_d/\sqrt{n}}$$

where $\bar{d} = \sum_{i=1}^{n} \frac{d_i}{n}$ and $S_d = \sqrt{\frac{\sum_{i=1}^{n}(d_i-\bar{d})^2}{n-1}}$, has a Student's t-distribution with $n-1$ degrees of freedom, and the statistic

$$F = T^2$$

has an F-distribution with 1 and $n-1$ degrees of freedom. However, if the $\{\gamma_i\}$ follow a stationary first order autoregressive Gaussian process

$$\gamma_i = \rho\gamma_{i-1} + e_i$$

then the distributions of T and F are unknown.

Chapter II surveys the literature dealing with the effects of dependent data upon the distributions of the T and F statistics. Various types of dependence are considered.

In Chapter III we examine the distribution of the F statistic when the data are sampled from a stationary first order autoregressive

Gaussian process. For a sample of size $n = 2$, the exact probability density of F is shown to be

$$\phi(F) = \frac{\sqrt{(1-\rho)(1+\rho)}}{\pi[1 + \rho + (1-\rho)F]\sqrt{F}} \quad , \quad 0 < F < \infty .$$

It is also shown that for large n, the probability density of F may be approximated by

$$g(F) = \sqrt{\frac{n}{(n-1)}} \; \frac{K_n \beta_1^{n/2}(1-\rho\beta_1)}{\sqrt{F} \; (1-\rho^2\beta_1)}\{1 + O(n^{-1})\} \quad , \quad 0 < F < \infty$$

with

$$\beta_1 = \frac{(1+\rho^2) + F(1-\rho)^2/(n-1) - \sqrt{[(1+\rho^2) + F(1-\rho)^2/(n-1)]^2 - 4\rho^2}}{2\rho^2}$$

and

$$K_n = \frac{(1-\rho)\Gamma(n/2)}{\sqrt{n\pi}\;\Gamma[(n-1)/2]}\left[F\left(\frac{3}{2}, \frac{n-1}{2}; \frac{n}{2}; \rho^2\right) - \frac{\rho(n-1)}{n} F\left(\frac{3}{2}, \frac{n+1}{2}; \frac{n+2}{2}; \rho^2\right)\right.$$

$$- \frac{\rho^2(n-1)(n+1)}{n(n+2)} F\left(\frac{3}{2}, \frac{n+3}{2}; \frac{n+4}{2}; \rho^2\right)$$

$$\left. + \frac{\rho^3(n-1)(n+2)(n+3)}{n(n+2)(n+4)} F\left(\frac{3}{2}, \frac{n+5}{2}; \frac{n+6}{2}; \rho^2\right)\right]^{-1}$$

where $F(\alpha,\beta;\gamma;z)$ is the hypergeometric function. Finally, large sample approximations for the moments of F are obtained.

CHAPTER II

LITERATURE REVIEW

If $\underline{X} = (X_1, X_2, \ldots, X_n)'$ is a vector of n independent identically distributed normal random variables, each with mean μ and variance σ^2, and if

$$\bar{X} = \frac{\sum_{i=1}^{n} X_i}{n} \quad \text{and} \quad S = \sqrt{\frac{\sum_{i=1}^{n} (X_i - \bar{X})^2}{n-1}}$$

then the statistic

$$T = \frac{\bar{X} - \mu}{S/\sqrt{n}}$$

has a Student's t-distribution with $n-1$ degrees of freedom, and

$$F = T^2 = \frac{n(\bar{X}-\mu)^2}{S^2}$$

has an F-distribution with 1 and $n-1$ degrees of freedom.

Independence of the $\{X_i\}$ is a major assumption used in determining the distributions of the T and F statistics. Thus it is of considerable importance to examine the sensitivity of these statistics to conditions of dependence. In this section we review the relevant literature.

Let $\{\theta_n\}$ be a sequence of statistics such that

$$\frac{\theta_n - E(\theta_n)}{\sqrt{V(\theta_n)}}$$

is asymptotically distributed as a standard normal random variable, and consider testing the null hypothesis H_0: $E(\theta_n) = 0$. The critical point K_α corresponding to a right-tailed test of size α is determined by the relation

$$1 - \Phi(K_\alpha) = \alpha ,$$

where Φ is the standard normal distribution function. If we denote by D the asymptotic variance of θ_n in the case of dependent observations from a stationary process, its value in the case of independent identically distributed observations by V, and the ratio D/V by τ , then, as shown by Gastwirth and Rubin (1971), the critical region determined under the assumption of independent observations, namely

$$\frac{\theta_n - E(\theta_n)}{\sqrt{V(\theta_n)}} \geqslant K_\alpha ,$$

has approximate probability

$$\alpha^* = 1 - \Phi(K_\alpha \tau^{-1/2}) .$$

Following these authors, we define α^* to be the asymptotic level of the right-tailed test based on θ_n when the observations are dependent.

Scheffé (1959) considered the model where $\underline{X} = (X_1, X_2, \ldots, X_n)'$ has a multivariate normal distribution with mean vector $\underline{\mu} = (\mu, \mu, \ldots, \mu)'$

and covariance matrix $\underset{-}{\Sigma}$, where

$$\underset{-}{\Sigma} = \sigma^2 \begin{bmatrix} 1 & \rho & & & & (0) & \\ \rho & 1 & \rho & & & & \\ & \rho & 1 & \rho & & & \\ & & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot & \\ & (0) & & & \rho & 1 & \rho \\ & & & & & \rho & 1 \end{bmatrix} .$$

To ensure that $\underset{-}{\Sigma}$ is positive definite, he imposed the necessary and sufficient restriction

$$-\left\{2\left[\cos \frac{\pi}{(n+1)}\right]\right\}^{-1} < \rho < \left\{2 \cos\left[\frac{\pi}{(n+1)}\right]\right\}^{-1} .$$

Under these conditions, he showed that T is asymptotically $N(0, 1+2\rho)$, and that the asymptotic level of the right-tailed T-test of H_0: $\mu = 0$ is given by

$$\alpha^* = 1 - \Phi(K_\alpha(1+2\rho)^{-1/2}) .$$

Let $\underline{X} = (X_1,X_2,\ldots,X_n)'$ have a multivariate normal distribution with mean vector $\underline{0} = (0,0,\ldots,0)'$ and covariance matrix $\underset{-}{\Sigma}$. Let $\underset{-}{\lambda} = \underset{-}{\Sigma}^{-1}$ and let λ_{ij} denote the element in the i^{th} row and j^{th} column of $\underset{-}{\lambda}$. Hotelling (1961) demonstrated that for large t, the statistic

$$R_n = \frac{n^{n/2}|\lambda|^{1/2}}{(\sum \sum \lambda_{ij})^{n/2}}$$

approximates the ratio

$$\frac{P(T>t \mid \underline{\Sigma}^{-1} = \underline{\lambda})}{P(T>t \mid \underline{\Sigma}^{-1} = \underline{I})} ,$$

where $\underline{I}$ is the $n \times n$ identity matrix. For the covariance matrix

$$\underline{\Sigma} = \begin{bmatrix} 1 & \rho & \rho^2 & \cdots & \rho^{n-1} \\ \rho & 1 & \rho & \cdots & \rho^{n-2} \\ \vdots & \vdots & \vdots & & \vdots \\ \rho^{n-1} & \rho^{n-2} & \rho^{n-3} & \cdots & 1 \end{bmatrix}$$

he found R_n to be

$$R_n = \frac{(1-\rho)^{1/2}}{(1-\rho)^n[1+2\rho(1-\rho)^{-1}n^{-1}]^{n/2}}$$

while for

$$\underline{\Sigma} = \begin{bmatrix} 1 & \rho & \rho & \cdots & \rho \\ \rho & 1 & \rho & \cdots & \rho \\ \rho & \rho & 1 & \cdots & \rho \\ \vdots & \vdots & \vdots & & \vdots \\ \rho & \rho & \rho & \cdots & 1 \end{bmatrix} ,$$

with $\rho > -(n-1)^{-1}$, he showed that

$$R_n = \left[1 + \frac{n\rho}{1-\rho}\right]^{(n-1)/2} .$$

In this latter case Ali (1973) proved that the exact probability density function of T is

$$g(t) = \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2})(\pi(n-1))^{1/2}(1+\frac{np}{1-\rho})^{1/2}} \cdot \frac{1}{[(1+\frac{n\rho}{1-\rho})^{-1}\frac{t^2}{(n-1)}+1]^{n/2}} \ .$$

He also showed that the tails of the t-distribution for $\rho > 0$ are thicker than those of the standard t-distribution $(\rho=0)$, while for $\rho < 0$ the tails are thinner than for the standard case.

Gastwirth and Rubin (1971) examined the behaviour of the asymptotic levels of the sign, Wilcoxon, and t-tests for various dependent processes. They showed:

i) If $X_1, X_2, \ldots, X_n$ are from a completely regular Gaussian process whose autocorrelations satisfy $\sum \rho_k < \infty$ and $\rho_k \geqslant 0$ for all k, then the level of the sign test is less sensitive to the dependence than the level of the Wilcoxon test which is less sensitive that of the t-test. In each case the level exceeds the corresponding value for independent observations.

ii) If $X_1, X_2, \ldots, X_n$ are from a first-order autoregressive Gaussian process with negative autocorrelation coefficient ρ, then the level of the sign test changes less with the dependence than the level of the Wilcoxon test which changes less than that of the t-test.

Albers (1978) studied the model in which the random vector $\underline{X} = (X_1, X_2, \ldots, X_n)'$ has a multivariate normal distribution with mean vector $\underline{\mu} = (\mu, \mu, \ldots, \mu)'$, and the $\{X_i\}$ are m-dependent; that is, the covariance matrix $\underline{\Sigma}$ has elements

$$\begin{aligned} \sigma_{ij} = \mathrm{Cov}(X_i, X_j) &= \sigma^2 \rho_{|i-j|} \quad , \quad 1 \leqslant |i-j| \leqslant m \\ &= 0 \quad , \quad |i-j| > m \ , \end{aligned}$$

for $i,j = 1,\ldots,n$, where m is a positive integer and ρ_k, $k = 1,\ldots,m$ are constants such that $\underline{\Sigma}$ is positive definite and

$$1 + 2 \sum_{k=1}^{m} \rho_k > 0 \ .$$

Under the given model, he showed that T^* is asymptotically

$$N(0,\ 1 + 2 \sum_{k=1}^{m} \rho_k (1 - \tfrac{k}{n}))$$

and that the asymptotic level of the right-tailed T-test of H_0: $\mu = 0$ is

$$\alpha^* = 1 - \Phi(K_\alpha \{1 + 2 \sum_{k=1}^{m} \rho_k (1 - \tfrac{k}{n})\}^{-1/2}) + o(1) \ .$$

This same paper also dealt with the situation in which the $\{X_i\}$ are from a stationary autoregressive process of order m; that is,

$$\sum_{k=0}^{m} a_k(X_{i-k}-\mu) = z_i \quad , \quad i = m+1,\ldots,n$$

where $z_{m+1},\ldots,z_n$ are independent $N(0,\gamma^2)$ random variables, $a_0 = 1$, and $a_1,\ldots,a_m$ are constants such that all roots of the equation

$$\sum_{k=0}^{m} a_k w^{m-k} = 0$$

lie inside the unit circle. Under these conditions he showed that T is asymptotically

$$N(0, \sum_{k=0}^{m} a_k\rho_k \Big/ (\sum_{k=0}^{m} a_k)^2 + O(\frac{m}{n})) \ ,$$

where ρ_k is the autocorrelation coefficient at lag k, for $k = 0,1,\ldots,m$.

For each model, Albers proposed a robust modification of the t-test. He demonstrated that under independence both of the proposed tests would require, asymptotically, mk_α^2 additionally observations to obtain the power of the t-test.

CHAPTER III

THE ASYMPTOTIC DISTRIBUTION OF THE SQUARE OF THE T STATISTIC FOR A FIRST ORDER AUTOREGRESSIVE GAUSSIAN PROCESS

3.1 First Order Autoregressive Processes

Consider a finite subset of time points $t_1 < t_2 < \ldots < t_k$ and let $F[X(t_1),X(t_2),\ldots,X(t_k)]$ denote the joint distribution function of the random variables $X(t_1)$, $X(t_2),\ldots,X(t_k)$. A time series or stochastic process is said to be strictly stationary if

$$F[X(t_1),X(t_2),\ldots,X(t_k)] = F(x(t_1-\tau),X(t_2-\tau),\ldots,X(t_k-\tau)]$$

for all nonempty finite subsets $t_1 < t_2 < \ldots < t_k$ and all τ ; that is, if the joint distribution is independent of the time origin.

A process is defined to be weakly stationary if for all t, t^*, and τ

i) $E[X(t)] = E[X(t-\tau)]$,

ii) $V[X(t)] = V[X(t-\tau)]$, and

iii) $Cov[X(t),X(t^*)] = Cov[X(t-\tau),X(t^*-\tau)]$.

That is, for a weakly stationary process the mean and variance are independent of time, and the covariance is invariant under a change of time origin.

Consider the first order autoregressive process

$$X_t = \rho X_{t-1} + e_t \ , \qquad t = 0, \pm 1, \pm 2,\ldots \tag{3.1.1}$$

where the $\{e_t\}$ are uncorrelated random variables, each with mean 0 and variance σ^2. Following Fuller (1976), it may be seen by repeated application of (3.1.1) that

$$\begin{aligned} X_t &= \rho^2 X_{t-2} + \rho e_{t-1} + e_t \\ &= \rho^3 X_{t-3} + \rho^2 e_{t-2} + \rho e_{t-1} + e_t \\ &\;\vdots \\ &= \rho^k X_{t-k} + \sum_{i=0}^{k-1} \rho^i e_t \,. \end{aligned}$$

Assume that $|\rho| < 1$. Then

$$\begin{aligned} X_t &= \lim_{k\to\infty} \left(\rho^k X_{t-k} + \sum_{i=0}^{k-1} \rho^i e_t\right) \\ &= \sum_{i=0}^{\infty} \rho^i e_t \end{aligned}$$

almost surely, and thus

$$E(X_t) = E\Big(\sum_{i=0}^{\infty} \rho^i e_t\Big) = 0 \tag{3.1.2}$$

and

$$\begin{aligned} \mathrm{Cov}(X_t, X_{t+h}) &= E[(X_t - E(X_t))(X_{t+h} - E(X_{t+h}))] \\ &= E(X_t X_{t+h}) = E\left[\left(\sum_{i=0}^{\infty} \rho^i e_{t-i}\right)\left(\sum_{j=0}^{\infty} \rho^j e_{t+h-j}\right)\right] \\ &= \sigma^2 \sum_{i=0}^{\infty} \rho^i \rho^{i+h} = \sigma^2 \frac{\rho^h}{1-\rho^2} \,, \end{aligned}$$

almost surely. Thus, for $|\rho| < 1$ the first order autoregressive process (3.1.1) is weakly stationary. Suppose in addition that the $\{e_t\}$ are uncorrelated $N(0,\sigma^2)$ random variables, so that any finite subset of the $\{X_t\}$ has a multivariate normal distribution. Then, since a multivariate normal distribution is determined by its first two moments, the modified process is strictly stationary.

3.2 The Joint Distribution of $X_1, X_2, \ldots, X_n$

Let $X_1, X_2, \ldots, X_n$ be sample observations from the stationary first order autoregressive process

$$(3.2.1) \qquad X_t = \rho X_{t-1} + e_t \ , \qquad t = 0 \ , \pm 1 \ , \pm 2 \ , \ldots$$

where $|\rho| < 1$ and the $\{e_t\}$ are independent $N(0,1)$ random variables. From equation (3.1.2) we have

$$E(X_t) = 0 \quad \text{and} \quad \mathrm{Cov}(X_t, X_{t+h}) = \frac{\rho^h}{1-\rho^2} \ .$$

Thus $\underline{X} = (X_1, X_2, \ldots, X_n)'$ has a multivariate normal distribution with probability density

$$f(\underline{X}) = (2\pi)^{-n/2} |\underline{\Sigma}|^{-1/2} \exp\left[\frac{-1}{2} (\underline{X}' \underline{\Sigma}^{-1} \underline{X})\right]$$

where $\underline{\Sigma}$ is the $n \times n$ covariance matrix

$$\underline{\Sigma} = \begin{bmatrix} \frac{1}{1-\rho^2} & \frac{\rho}{1-\rho^2} & \cdots & \frac{\rho^{n-1}}{1-\rho^2} \\ \frac{\rho}{1-\rho^2} & \frac{1}{1-\rho^2} & \cdots & \frac{\rho^{n-2}}{1-\rho^2} \\ \vdots & \vdots & & \vdots \\ \frac{\rho^{n-1}}{1-\rho^2} & \frac{\rho^{n-2}}{1-\rho^2} & \cdots & \frac{1}{1-\rho^2} \end{bmatrix} .$$

The following method of evaluating $|\underline{\Sigma}|$ is due to Patton (1961).

$$|\underline{\Sigma}| = \begin{vmatrix} \frac{1}{1-\rho^2} & \frac{\rho}{1-\rho^2} & \cdots & \frac{\rho^{n-1}}{1-\rho^2} \\ \frac{\rho}{1-\rho^2} & \frac{1}{1-\rho^2} & \cdots & \frac{\rho^{n-2}}{1-\rho^2} \\ \vdots & \vdots & & \vdots \\ \frac{\rho^{n-1}}{1-\rho^2} & \frac{\rho^{n-2}}{1-\rho^2} & \cdots & \frac{1}{1-\rho^2} \end{vmatrix}_n$$

Multiplying the second column by ρ and subtracting from the first column, we obtain

$$|\underline{\Sigma}| = \begin{vmatrix} 1 & \frac{\rho}{1-\rho^2} & \cdots & \frac{\rho^{n-1}}{1-\rho^2} \\ 0 & \frac{1}{1-\rho^2} & \cdots & \frac{\rho^{n-2}}{1-\rho^2} \\ \vdots & \vdots & & \vdots \\ 0 & \frac{\rho^{n-2}}{1-\rho^2} & \cdots & \frac{1}{1-\rho^2} \end{vmatrix}_n = \begin{vmatrix} \frac{1}{1-\rho^2} & \frac{\rho}{1-\rho^2} & \cdots & \frac{\rho^{n-2}}{1-\rho^3} \\ \frac{\rho}{1-\rho^2} & \frac{1}{1-\rho^2} & \cdots & \frac{\rho^{n-3}}{1-\rho^2} \\ \vdots & \vdots & & \vdots \\ \frac{\rho^{n-2}}{1-\rho^2} & \frac{\rho^{n-3}}{1-\rho^2} & \cdots & \frac{1}{1-\rho^2} \end{vmatrix}_{n-1} .$$

Continuing in this fashion we find that

$$|\underline{\Sigma}| = \begin{vmatrix} \frac{1}{1-\rho^2} & \frac{\rho}{1-\rho^2} \\ \frac{\rho}{1-\rho^2} & \frac{1}{1-\rho^2} \end{vmatrix} = \frac{1}{1-\rho^2} .$$

We wish to determine an expression for $\underline{X}'\underline{\Sigma}^{-1}\underline{X}$. Following Routledge's (1972) derivation of $\underline{\Sigma}^{-1}$, we define the $n \times n$ matrix $\underline{J}$ by

$$\underline{J} = \begin{bmatrix} 1 & -\rho & & & & & \\ & 1 & -\rho & & & (0) & \\ & & 1 & -\rho & & & \\ & & & \cdot & \cdot & & \\ & & & & \cdot & \cdot & \\ & (0) & & & & 1 & -\rho \\ & & & & & & 1 \end{bmatrix} .$$

Then $\underline{\tilde{\Sigma}} = \underline{J}\underline{\Sigma}$ is the $n \times n$ lower triangular matrix

$$\underline{\tilde{\Sigma}} = \begin{bmatrix} 1 & & & & & \\ \rho & 1 & & & (0) & \\ \rho^2 & \rho & 1 & & & \\ \vdots & \vdots & \vdots & & & \\ \rho^{n-2} & \rho^{n-3} & \rho^{n-4} & \cdots & 1 & \\ \frac{\rho^{n-1}}{1-\rho^2} & \frac{\rho^{n-2}}{1-\rho^2} & \frac{\rho^{n-3}}{1-\rho^2} & \cdots & \frac{\rho}{1-\rho^2} & \frac{1}{1-\rho^2} \end{bmatrix} .$$

To obtain the $n \times n$ identity matrix $\underline{I}$ from $\underline{\tilde{\Sigma}}$ we perform the elementary column operations $-\rho C_2 + C_1$, $-\rho C_3 + C_2$,..., $-\rho C_n + C_{n-1}$, $(1-\rho^2)C_n$. Applying these same linear transformations to $\underline{I}$ we see that

$$\tilde{\underline{\Sigma}}^{-1} = \begin{bmatrix} 1 & & & & & (0) & \\ -\rho & 1 & & & & & \\ & -\rho & 1 & & & & \\ & & \cdot & \cdot & & & \\ & & & \cdot & \cdot & & \\ & (0) & & & -\rho & 1 & \\ & & & & & -\rho & 1-\rho^2 \end{bmatrix} .$$

Thus

$$\underline{\Sigma}^{-1} = \tilde{\underline{\Sigma}}^{-1}\underline{J}$$

$$= \begin{bmatrix} 1 & -\rho & & & & & \\ -\rho & 1+\rho^2 & -\rho & & & (0) & \\ & -\rho & 1+\rho^2 & -\rho & & & \\ & & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot & \\ & (0) & & & -\rho & 1+\rho^2 & -\rho \\ & & & & & -\rho & 1 \end{bmatrix} .$$

and

$$\underline{X}'\underline{\Sigma}^{-1}\underline{X} = X_1^2 + X_n^2 + (1+\rho^2)(X_2^2+\ldots+X_{n-1}^2)$$

$$- 2\rho(X_1X_2+X_2X_3+\ldots+X_{n-2}X_{n-1}+X_{n-1}X_n) .$$

3.3 The Exact Probability Density of F for a Sample of Size n = 2

In this section we determine the exact probability density of the F statistic for the case in which the sample consists of 2 observations.

Let X_1 and X_2 be sample observations from the stationary first order autoregressive process (3.2.1). Then X_1 and X_2 have a joint bivariate normal distribution with mean vector $\underline{0} = (0,0)'$, covariance matrix

$$\underline{\Sigma} = \frac{1}{1-\rho^2}\begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$$

and probability density

$$f(X_1,X_2) = \frac{\sqrt{1-\rho^2}}{2\pi} \exp\left[-\frac{1}{2}(X_1^2-2\rho X_1X_2+X_2^2)\right].$$

In addition we have

$$F = \frac{2\bar{X}^2}{S^2}$$

where

$$\bar{X} = \frac{X_1+X_2}{2} \text{ and } S^2 = \frac{(X_1-\bar{X})^2+(X_2-\bar{X})^2}{2-1} = \frac{(X_1-X_2)^2}{2}.$$

Thus

$$F = \frac{(X_1+X_2)^2}{(X_1-X_2)^2} = \frac{U^2}{V^2}$$

where we have introduced the transformation

(3.3.1) $$U = X_1 + X_2 \quad , \quad V = X_1 - X_2 \quad .$$

The inverse transformation of (3.3.1) is

$$X_1 = \frac{U+V}{2} \quad , \quad X_2 = \frac{U-V}{2}$$

with $-\infty < U < \infty$ and $-\infty < V < \infty$. The joint probability density of U and V is given by

$$g(u,v) = f[X_1(u,v),X_2(u,v)]|J| \ ,$$

where

$$J = \left|\frac{\partial(X_1,X_2)}{\partial(u,v)}\right| = \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix} = \frac{-1}{2} \ .$$

Thus

$$g(u,v) = \frac{\sqrt{1-\rho^2}}{4\pi} \exp\left\{\frac{-1}{2}\left[\left(\frac{u+v}{2}\right)^2 - 2\rho\left(\frac{u+v}{2}\right)\left(\frac{u-v}{2}\right) + \left(\frac{u-v}{2}\right)^2\right]\right\}$$

$$= \frac{\sqrt{1-\rho^2}}{4\pi} \exp\left\{\frac{-1}{2}\left[\frac{u^2(1-\rho)+v^2(1+\rho)}{2}\right]\right\} \quad ,$$

which is the probability density of a bivariate normal random vector $\underline{Y} = (U,V)'$ with mean $\underline{0} = (0,0)'$ and covariance matrix $\underline{\Sigma}_y$ with inverse

$$\underline{\Sigma}_y^{-1} = \begin{bmatrix} \frac{1-\rho}{2} & 0 \\ 0 & \frac{1+\rho}{2} \end{bmatrix} \quad .$$

By inspection we obtain

$$\underset{\sim}{\Sigma}_y = \begin{bmatrix} \frac{2}{1-\rho} & 0 \\ 0 & \frac{2}{1+\rho} \end{bmatrix} ,$$

and therefore U and V are independent normal random variables with

$$U \sim N\left(0, \frac{2}{1-\rho}\right) \quad \text{and} \quad V \sim N\left(0, \frac{2}{1+\rho}\right) .$$

It follows that

$$\sqrt{\frac{(1-\rho)}{2}}\, U \quad \text{and} \quad \sqrt{\frac{(1+\rho)}{2}}\, V$$

are independent standard normal random variables and

$$\frac{(1-\rho)}{2} U^2 \quad \text{and} \quad \frac{(1+\rho)}{2} V^2$$

are independent chi-square random variables, each with 1 degree of freedom. Thus the statistic

$$F^* = \frac{(1-\rho)}{2} U^2 \Big/ \left[\frac{(1+\rho)}{2} V^2\right]$$

$$= \frac{(1-\rho)}{(1+\rho)} F$$

has an F-distribution with $\nu_1 = 1$ and $\nu_2 = 1$ degrees of freedom.

Let ϕ and ϕ^* denote the probability densities of F and F^*, respectively. Then

$$\phi^*(F^*) = \frac{\Gamma[(\nu_1+\nu_2)/2](\nu_1/\nu_2)^{\nu_1/2}}{\Gamma(\nu_1/2)\Gamma(\nu_2/2)} \frac{F^{*\,\nu_1/2-1}}{(1+\nu_1 F^*/\nu_2)^{(\nu_1+\nu_2)/2}}$$

$$= \frac{\Gamma(1)}{\Gamma(1/2)\Gamma(1/2)} \frac{F^{*-1/2}}{(1+F^*)}$$

$$= \frac{1}{\pi(1+F^*)\sqrt{F^*}}$$

and

$$\phi(F) = \phi^*(F^*(F)) \left|\frac{dF^*}{dF}\right|$$

$$= \frac{1}{\pi[1+(1-\rho)F/(1+\rho)]\sqrt{(1-\rho)F/(1+\rho)}} \frac{(1-\rho)}{(1+\rho)} .$$

Thus the exact probability density of the F-statistic is

$$\phi(F) = \frac{\sqrt{(1-\rho)(1+\rho)}}{\pi[1+\rho+(1-\rho)F]\sqrt{F}} , \; 0 < F < \infty .$$

3.4 The Cramér-Geary Inversion Formula

Daniels (1956) outlined a proof of the Cramér-Geary inversion formula. The following derivation is based upon that presented by Patton (1961).

Consider a statistic of the form

$$r = \frac{c_1}{c_0} ,$$

where c_0 is almost surely positive. Let $f(c_0,c_1)$ denote the joint probability density of c_0 and c_1. We wish to obtain the distribution of r from $f(c_0,c_1)$.

The transformation

$$c_0 = c_0 \quad , \; c_1 = rc_0$$

has Jacobian

$$\left|\frac{\partial(c_0,c_1)}{\partial(c_0,r)}\right| = \begin{vmatrix} 1 & 0 \\ r & c_0 \end{vmatrix} = c_0 ,$$

so that the joint probability density of c_0 and r is

$$c_0 f(c_0,rc_0) .$$

Integration over c_0 yields the marginal density

$$h(r) = \int_0^\infty c_0 f(c_0,rc_0)dc_0 , \tag{3.4.1}$$

which is the probability density of r.

Let $M(T_0,T_1)$ denote the joint moment generating function of c_0 and c_1 . Then

$$M(T_0,T_1) = E\left(e^{T_0c_0+T_1c_1}\right) = \int_{-\infty}^{\infty}\int_0^{\infty} e^{T_0c_0+T_1c_1} f(c_0,c_1)dc_0dc_1 .$$

By the Fourier inversion formula, it follows that

$$f(c_0,c_1) = \frac{1}{(2\pi i)^2} \int \int M(T_0,T_1)e^{-(T_0c_0+T_1c_1)} dT_0 dT_1 \ .$$

The integration is along the imaginary axes in the T_0 and T_1 planes from $-i\infty$ to $+i\infty$, or along any allowable deformation of these paths; that is, along any paths from $\xi_1 - i\infty$ to $\xi_2 + i\infty$ such that no singularities of $M(T_0,T_1)$ lie on the new paths of integration, or between them and the imaginary axes. Substitution of rc_0 for c_1 gives

$$f(c_0,rc_0) = \frac{1}{(2\pi i)^2} \iint M(T_0,T_1)e^{-c_0(T_0+rT_1)} dT_0 dT_1 \ . \tag{3.4.2}$$

Under the linear transformation

$$u = T_0 + rT_1 \quad , \quad T_1 = T_1 \ ,$$

with Jacobian

$$\left|\frac{\partial(T_0,T_1)}{\partial(u,T_1)}\right| = \begin{vmatrix} 1 & -r \\ 0 & 1 \end{vmatrix} = 1 \ ,$$

equation (3.4.2) becomes

$$f(c_0,rc_0) = \frac{1}{(2\pi i)^2} \iint M(u-rT_1,T_1)e^{-c_0u} \, du dT_1 \ .$$

The integration of u is along a path in the u-plane corresponding to that of T_0 in the T_0-plane. Thus

$$\int_0^\infty f(c_0,rc_0)e^{c_0u}dc_0 = \int_0^\infty \left\{ \frac{1}{(2\pi i)^2} \iint M(u-rT_1,T_1)e^{-c_0u} du dT_1 \right\} e^{c_0u} dc_0$$

$$= \frac{1}{2\pi i}\int \left\{ \int_0^\infty \left[\frac{1}{2\pi i} \int M(u-rT_1,T_1)e^{-c_0u} du \right] e^{c_0u} dc_0 \right\} dT_1$$

By the Fourier inversion formula,

$$M(u-rT_1,T_1) = \int_0^\infty \left[\frac{1}{2\pi i} \int M(u-rT_1,T_1)e^{-c_0u} du \right] e^{c_0u} \, dc_0 \; ,$$

so that

$$\int_0^\infty f(c_0,rc_0)e^{c_0u} \, dc_0 = \frac{1}{2\pi i} \int M(u-rT_1,T_1)dT_1 \; . \tag{3.4.3}$$

When differentiation under the integral sign with respect to u is permissible, we have

$$\int_0^\infty f(c_0,rc_0)c_0 e^{c_0u} dc_0 = \frac{1}{2\pi i} \int \frac{\partial}{\partial u}\left[M(u-rT_1,T_1)\right] dT_1 \; .$$

Letting $u = 0$, we obtain, by equation (3.4.1)

$$h(r) = \frac{1}{2\pi i} \int \frac{\partial}{\partial u} \left[M(u-rT_1,T_1)\right]\Big|_{u=0} \, dT_1 \; .$$

Often it is desirable to transform T_1 to another variable z by $T_1 = T_1(z,u)$, where $T_1(z,0)$ maps the T_1-plane onto some region of the z-plane. In this case

$$M(u-rT_1,T_1)dT_1 = M[u-rT_1(z,u),T_1(z,u)] \, \frac{\partial T_1(z,u)}{\partial z} \, dz$$

which, for notational ease, we express as

$$M(u-rT_1,T_1)\,\frac{\partial T_1}{\partial z}\,dz\ .$$

From equation (3.4.3) it follows that

$$\int_0^\infty f(c_0,rc_0)e^{c_0u}\,dc_0 = \frac{1}{2\pi i}\int M(u-rT_1,T_1)\,\frac{\partial T_1}{\partial z}\,dz\ .$$

The integration is along the transformed path in the z-plane. Once again differentiating with respect to u under the integral sign and letting $u = 0$, we obtain

$$h(r) = \frac{1}{2\pi i}\int \frac{\partial}{\partial u}\left[M(u-rT_1,T_1)\,\frac{\partial T_1}{\partial z}\right]\Bigg|_{u=0} dz\ . \tag{3.4.4}$$

This equation is known as the Cramér-Geary inversion formula and will be used in the work to follow.

3.5 The Joint Moment Generating Function of c_0 and c_1

Let $X_1, X_2, \ldots, X_n$ be sample observations from the stationary first order autoregressive Gaussian process (3.2.1) and let

$$F = \frac{n\bar{X}^2}{S^2} ,$$

where*

$$\bar{X} = \frac{\sum X_i}{n} \quad \text{and} \quad S^2 = \frac{\sum(X_i - \bar{X})^2}{n-1} = \frac{\sum X_i^2 - (\sum X_i)^2/n}{n-1} .$$

Then

$$F = \frac{(n-1)}{n} r ,$$

where

$$r = \frac{n(\sum X_i)^2}{n\sum X_i^2 - (\sum X_i)^2} .$$

Let g and h denote the probability densities of F and r, respectively. Then g and h are related by

(3.5.1) $$g(F) = h(r)\left|\frac{dr}{dF}\right| = \frac{n}{(n-1)}\, h\left[\frac{nF}{(n-1)}\right]$$

and

(3.5.2) $$h(r) = g(f)\left|\frac{dF}{dr}\right| = \frac{(n-1)}{n}\, g\left[\frac{(n-1)}{n}\, r\right] .$$

* All summations in section 3.5 are from $i = 1$ to $i = n$ unless otherwise indicated.

We may express r as the ratio

$$r = \frac{c_1}{c_0} ,$$

where

$$c_0 = (n-1)(X_1^2+X_2^2+\ldots+X_n^2) - 2(X_1X_2+X_1X_3+\ldots+X_{n-1}X_n)$$

and

$$c_1 = n(X_1^2+X_2^2+\ldots+X_n^2) + 2n(X_1X_2+X_1X_3+\ldots+X_{n-1}X_n) .$$

We shall use the Cramér-Geary inversion technique to obtain an asymptotic approximation of $h(r)$, from which we shall approximate $g(F)$. In this section we derive an expression for $M(T_0,T_1)$, the joint moment generating function of c_0 and c_1.

By definition,

$$\text{(3.5.3)} \qquad M(T_0,T_1) = E(e^{T_0c_0+T_1c_1})$$

$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{T_0c_0+T_1c_1} \frac{1}{(2\pi)^{n/2}|\underline{\Sigma}|^{1/2}} \exp\left[-\frac{1}{2}(\underline{X}'\underline{\Sigma}^{-1}\underline{X})\right] dX_1,\ldots,dX_n$$

$$= \frac{(1-\rho^2)^{1/2}}{(2\pi)^{n/2}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{T_0c_0+T_1c_1 - \frac{1}{2}(\underline{X}'\underline{\Sigma}^{-1}\underline{X})} dX_1,\ldots,dX_n$$

where

$$\underline{X}'\underline{\Sigma}^{-1}\underline{X} = X_1^2 + X_n^2 + (1+\rho^2)(X_2^2+\ldots+X_{n-1}^2)$$
$$- 2\rho(X_1X_2+X_2X_3+\ldots+X_{n-2}X_{n-1}+X_{n-1}X_n) .$$

Since

$$T_0c_0 + T_1c_1 - \frac{1}{2}\,\underline{X}\,\underline{\Sigma}^{-1}\underline{X}$$

$$= (X_1^2+X_n^2)[(n-1)T_0+nT_1-\tfrac{1}{2}] + (X_2^2+\ldots+X_{n-1}^2)[(n-1)T_0+nT_1 - \tfrac{1}{2}(1+\rho^2)]$$

$$+ (X_1X_2+X_2X_3+\ldots+X_{n-2}X_{n-1}+X_{n-1}X_n)[-2T_0+2nT_1+\rho]$$

$$+ (X_1X_3+X_1X_4+\ldots+X_1X_n+X_2X_4+X_2X_5+\ldots+X_2X_n+\ldots+X_{n-2}X_n)[-2T_0+2nT_1] ,$$

we have that

$$(3.5.4) \qquad T_0c_0 + T_1c_1 - \tfrac{1}{2}(\underline{X}'\,\underline{\Sigma}^{-1}\underline{X}) = \tfrac{-1}{2}(\underline{X}'\,\underline{B}\underline{X}) ,$$

where $\underline{B}$ is the n×n matrix

$$\underline{B} = \begin{bmatrix} f & b & & & & & \\ b & a & b & & (c) & & \\ & b & a & b & & & \\ & & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot & \\ & (c) & & & b & a & b \\ & & & & & b & f \end{bmatrix}$$

with

$$a = 1 + \rho^2 - 2(n-1)T_0 - 2nT_1 ,$$

$$b = 2T_0 - 2nT_1 - \rho ,$$

$$c = 2T_0 - 2nT_1 , \text{ and}$$

$$f = 1 - 2(n-1)T_0 - 2nT_1 .$$

Combining equations (3.5.3) and (3.5.4), we obtain

$$M(T_0,T_1) = \frac{(1-\rho^2)^{1/2}}{(2\pi)^{n/2}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-\frac{1}{2}(\underline{X}'\underline{B}\underline{X})} dX_1,\ldots,dX_n ,$$

which, by the theory of the multivariate normal distribution, reduces to

$$M(T_0,T_1) = \frac{(1-\rho^2)^{1/2}}{|\underline{B}|^{1/2}} . \tag{3.5.5}$$

From equation (I.23) of Appendix I we have

$$|\underline{B}| = \frac{u^{n-1}}{(u-v)(2e+d)^2}\Big\{[u+(f-a)]^2[(2e+d)^2-c((u-v)-n(2e+d))]$$

$$+c(2e+d)[u^2-(f-a)^2]\Big\}$$

$$- \frac{v^{n-1}}{(u-v)(2e+d)^2}\Big\{[v+(f-a)]^2[(2e+d)^2+c((u-v)+n(2e+d))]$$

$$+c(2e+d)[v^2-(f-a)^2]\Big\}$$

$$+ \frac{2c(-e)^{n-1}}{(2e+d)^2}[e-(f-a)]^2$$

where u and v are roots of the equation

$$x^2 - dx + e^2 = 0$$

with

$$d = a - c = 1 + \rho^2 - 2nT_0$$

and

$$e = b - c = -\rho .$$

This expression for $|\underline{B}|$ is symmetric with respect to u and v.

Also, the roots of

$$x^2 - dx + e^2 = 0$$

are

$$\frac{d \pm \sqrt{d^2-4e^2}}{2},$$

so that

(3.5.7) $$uv = \frac{d^2 - (d^2-4e^2)}{4} = e^2 .$$

3.6 An Asymptotic Approximation for $|B|$

Setting $u = z^{-1}$ we obtain, from equations (3.5.6) and (3.5.7),

$$z^{-2} - dz^{-1} + e^2 = 0 \tag{3.6.1}$$

and

$$v = e^2 z = \rho^2 z \ .$$

Let $\mu = T_0 + rT_1$. Then from equations (3.6.1) and (3.5.6) we have

$$z^{-1} + e^2 z = d$$

or

$$z^{-1} + \rho^2 z = 1 + \rho^2 - 2nT_0 = 1 + \rho^2 - 2n\mu + 2nrT_1 \ . \tag{3.6.2}$$

When $\mu = 0$ equation (3.6.2) becomes

$$z^{-1} + \rho^2 z = 1 + \rho^2 + 2nrT_1 \tag{3.6.3}$$

and

$$T_1 = \frac{1}{2nr}\left[z^{-1} + \rho^2 z - 1 - \rho^2\right] = \frac{1}{2nr}\left[\frac{1-z}{z} - \rho^2(1-z)\right]$$

or

$$T_1 = \frac{(1-z)}{2nr}\left[\frac{1}{z} - \rho^2\right]. \tag{3.6.4}$$

The mapping (3.6.3) consists of the mappings

A: $z^{-1} + \rho^2 z = w$

and

B: $w = 1 + \rho^2 + 2nrT_1$.

A is a simple modification of the Joukowski transformation

$$z^{-1} + z = w ,$$

which is discussed in Appendix II. By arguments similar to those presented in Appendix II, it may be shown that A maps the w-plane, cut along the real axis from $-2|\rho|$ to $2|\rho|$, inclusive, onto the interior of the circle $|z| = \frac{1}{|\rho|}$.

Under the second mapping, if $w = -2|\rho|$ then

$$T_1 = -\frac{(1+2|\rho|+\rho^2)}{2nr} = -\frac{(1+|\rho|)^2}{2nr} ,$$

and if $w = 2|\rho|$ then

$$T_1 = -\frac{(1-2|\rho|+\rho^2)}{2nr} = -\frac{(1-|\rho|)^2}{2nr} .$$

Thus, B maps the T_1-plane, cut along the real axis from $-\frac{(1+|\rho|)^2}{2nr}$ to $-\frac{(1-|\rho|)^2}{2nr}$, onto the cut w-plane. For $|\rho| < 1$ and all positive n and r, the endpoints of the cut in the T_1-plane are negative.

Of particular interest is the transformed path of integration, that is, the path in the z-plane into which the imaginary axis in the T_1-plane is transformed. Let $z = x + iy$. Then equation (3.6.4) becomes

$$T_1 = \frac{(1-x-iy)}{2nr}\left(\frac{x-iy}{x^2+y^2} - \rho^2\right)$$

$$= \frac{1}{2nr}\Big[(1-x)-iy\Big]\left[\frac{x}{x^2+y^2} - \rho^2 - \frac{iy}{x^2+y^2}\right],$$

so that

$$\mathrm{Re}(T_1) = \frac{1}{2nr}\left[(1-x)\left(\frac{x}{x^2+y^2} - \rho^2\right) - \frac{y^2}{x^2+y^2}\right] .$$

The imaginary axis in the T_1-plane has equation $\mathrm{Re}(T_1) = 0$ and thus, on the transformed path of integration

$$0 = \frac{x}{x^2+y^2} - \rho^2 - \frac{x^2}{x^2+y^2} + x\rho^2 - \frac{y^2}{x^2+y^2}$$

$$= x\left(\frac{1}{x^2+y^2} + \rho^2\right) - (1+\rho^2) .$$

It follows that

(3.6.5) $$x^2 + y^2 = \frac{x}{1+\rho^2(1-x)}$$

and

$$y = \pm\sqrt{\frac{x}{1+\rho^2(1-x)} - x^2} = \pm\sqrt{\frac{x(1-x)-\rho^2(1-x)x^2}{1+\rho^2(1-x)}}$$

$$= \pm\sqrt{\frac{x(1-x)(1-\rho^2 x)}{1+\rho^2(1-x)}} .$$

Since y is real, we have

$$\frac{x(1-x)(1-\rho^2 x)}{1+\rho^2(1-x)} \geqslant 0 .$$

But $|\rho| < 1$ and $|z| < \frac{1}{|\rho|}$, and so we must have $0 \leqslant x \leqslant 1$. We also see that for any given x, $|y|$ is a decreasing function of ρ^2

which is maximized when $\rho = 0$. In this case $|y| = \sqrt{x(1-x)}$, which has maximum value 1/2 when $x = 1/2$. Thus $0 \leqslant |y| \leqslant 1/2$ on the transformed path of integration. From equation (3.6.5) it follows that

$$|z| = \sqrt{x^2+y^2} = \sqrt{\frac{x}{1+\rho^2(1-x)}} \quad , \; 0 \leqslant x \leqslant 1$$

which is an increasing function of x and has maximum value 1 when $x = 1$. Thus, $|z| \leqslant 1$ on the path of integration. Additionally, we see from equation (3.6.4) that as $T_1 \to 0$, $z \to 1$ and as $T_1 \to \pm i\infty$, $z \to 0$. Therefore as T_1 traverses the imaginary axis from $- i\infty$ to 0, z moves clockwise along a path in quadrant I from (0,0) to (1,0), and as T_1 goes along the imaginary axis from 0 to $+ i\infty$, z travels clockwise in quadrant IV from (1,0) to (0,0).

Substituting $e = -\rho$, $u = z^{-1}$, and $v = \rho^2 z$ into the expression (3.5.6) for $|\underline{B}|$, we see that the second term (that is, the term with v^{n-1}) involves a factor $\rho^{2(n-1)} z^{n-1}$ and the third term (that is, the term with $(-e)^{n-1}$) involves a factor ρ^{n-1} . Since $|\rho| < 1$ and $|z| \leqslant 1$ *, it follows that both terms will make contributions which are exponentially small for large n. Furthermore, in the case $\rho = 0$ they make no contribution at all. Thus we may neglect these terms and approximate $|\underline{B}|$ by

* In fact $|z| < 1$ on the transformed path of integration except at the point $z = (1,0)$.

$$(3.6.6)\quad |\underline{B}| \approx \frac{u^{n-1}}{(u-v)(2e+d)^2}\Big\{[u+(f-a)]^2[(2e+d)^2-c((u-v)-n(2e+d))]$$

$$+ c(2e+d)[u^2-(f-a)^2]\Big\}$$

$$= \frac{u^{n-1}(u+f-a)}{(u-v)(2e+d)^2}\Big\{[u+(f-a)][(2e+d)^2-c((u-v)-n(2e+d))]$$

$$+ c(2e+d)[u-f+a]\Big\} .$$

In the special case $(\rho=0)$, (3.6.6) is an exact expression for $|\underline{B}|$.

3.7 The Special Case (ρ=0)

Later we shall use the Cramér-Geary inversion formula to obtain the asymptotic distribution of the r statistic. To illustrate the method, we derive the exact distribution of r for the special case (ρ=0).

Let $\rho = 0$. From equations (3.5.4), (3.5.6), and (3.6.6) we have

$$|\underline{B}| = \frac{u^{n-1}(u+f-a)}{(u-v)(2e+d^2)}\Big\{[u+(f-a)][(2e+d^2)-c((u-v)-n(2e+d))]$$
$$+ c(2e+d)[u-f+a]\Big\}$$

where $u = z^{-1}$, $f - a = -\rho^2 = 0$, $v = \rho^2 z = 0$, $e = -\rho = 0$, and $d = z^{-1} + \rho^2 z = z^{-1}$. Thus

$$|\underline{B}| = \frac{z^{-(n-1)}(z^{-1})}{z^{-1}(z^{-2})}\Big\{z^{-1}[z^{-2}-c(z^{-1}-nz^{-1})] + c(z^{-1})(z^{-1})\Big\}$$

$$= \frac{1}{z^{n-3}}\Big\{\frac{1}{z^3} - \frac{c}{z^2} + \frac{nc}{z^2} + \frac{c}{z^2}\Big\}$$

or

(3.7.1) $$|\underline{B}| = \frac{1}{z^n}\{1 + ncz\} .$$

Solving for T_1 in equation (3.6.2) we obtain

(3.7.2) $$T_1 = \frac{1}{2nr}[z^{-1}-1+2n\mu] .$$

From (3.5.4) we have

$$c = 2T_0 - 2nT_1 = 2\mu - 2(n+r)T_1$$

so that

$$c = 2\mu - \frac{2(n+r)}{2nr}[z^{-1}-1+2n\mu]$$

$$= 2\mu - \frac{2(n+r)\mu}{r} - \frac{(n+r)}{nr} \frac{(1-z)}{z}$$

$$= \frac{-2n\mu}{r} - \frac{(n+r)}{nr} \frac{(1-z)}{z} .$$

It follows from equation (3.7.1) that

$$|\underline{B}| = \frac{1}{z^n}\left\{1 - \frac{2n^2\mu z}{r} - \frac{(n+r)}{r}(1-z)\right\} ,$$

and so, by equation (3.5.5),

$$M(T_0,T_1) = \frac{(1-\rho^2)^{1/2}}{|\underline{B}|^{1/2}} = \frac{1}{|\underline{B}|^{1/2}}$$

$$= z^{n/2}\left\{1 - \frac{2n^2\mu z}{r} - \frac{(n+r)}{r}(1-z)\right\}^{-1/2} .$$

Differentiating both sides of equation (3.7.2) wih respect to z we obtain

$$\frac{\partial T_1}{\partial z} = \frac{-1}{2nrz^2} .$$

so that

$$M(T_0,T_1)\frac{\partial T_1}{\partial z} = \frac{-z^{(n-4)/2}}{2nr}\left\{1 - \frac{2n^2\mu z}{r} - \frac{(n+r)}{r}(1-z)\right\}^{-1/2}$$

and

$$\frac{\partial}{\partial\mu} M(T_0\ ,T_1)\ \frac{\partial T_1}{\partial z}\bigg|_{\mu=0}$$

$$= \frac{-z^{(n-4)/2}}{2nr}\left\{1 - \frac{2n^2\mu z}{r} - \frac{(n+r)}{r}\ (1-z)\right\}^{-3/2}\left(\frac{-1}{2}\right)\left(\frac{-2n^2 z}{r}\right)\bigg|_{\mu=0}$$

$$= \frac{-nz^{(n-2)/2}}{2r^2}\left\{1 - \frac{(n+r)}{r}\ (1-z)\right\}^{-3/2}$$

$$= \frac{-nz^{(n-2)/2}}{2r^2}\left\{\frac{(n+r)}{r}\left[\frac{r}{(n+r)} - 1 + z\right]\right\}^{-3/2}$$

$$= \frac{-\ nz^{(n-2)/2}}{2\sqrt{r}(n+r)^{3/2}}\left\{z - \frac{r}{(n+r)}\right\}^{-3/2}$$

$$= -\frac{z^{(n-2)/2}}{2\sqrt{nr}(1+r/n)^{3/2}}\left\{z - \frac{n}{(n+r)}\right\}^{-3/2} \quad .$$

Thus

(3.7.3)
$$h(r) = \frac{1}{2\pi i}\int \frac{\partial}{\partial\mu}\left[M(\mu-rT_1,T_1)\ \frac{\partial T_1}{\partial z}\right]\bigg|_{\mu=0} dz$$

$$= -\frac{1}{4\pi i\sqrt{nr}(1+r/n)^{3/2}}\int_\Gamma z^{(n-2)/2}\left\{z - \frac{n}{(n+r)}\right\}^{-3/2} dz$$

$$= -\frac{1}{4\pi i\sqrt{nr}(1+r/n)^{3/2}}\int_\Gamma z^{(n-2)/2}(z-\beta)^{-3/2}dz \ ,$$

where $z = x + iy$, $0 < \beta = n/(n+r) < 1$, and Γ is the path of integration

$$\Gamma:\quad |y| = \sqrt{x(1-x)}\ ,\qquad 0 \leqslant x \leqslant 1$$

which is a circle of radius 1/2 centred at (1/2,0).

For $\rho = 0$, we know that the F statistic has an F distribution with $\nu_1 = 1$ and $\nu_2 = n - 1$ degrees of freedom. It follows that F has probability density

$$g(F) = \frac{\Gamma[(\nu_1+\nu_2)/2](\nu_1/\nu_2)^{\nu_1/2}}{\Gamma(\nu_1/2)\Gamma(\nu_2/2)} \frac{F^{\nu_1/2-1}}{(1+\nu_1 F/\nu_2)^{(\nu_1+\nu_2)/2}}$$

$$= \frac{\Gamma(n/2)[1/(n-1)]^{1/2}}{\Gamma(1/2)\Gamma[(n-1)/2)]} \frac{F^{-1/2}}{[1+F/(n-1)]^{n/2}}$$

$$= \frac{\Gamma(n/2)}{\sqrt{(n-1)\pi F}\,\Gamma[(n-1)/2]} \frac{1}{[1+F/(n-1)]^{n/2}} .$$

From equation (3.5.2) we have

$$\text{(3.7.4)}\quad h(r) = \frac{(n-1)}{n}\, g\left[\frac{(n-1)}{n}\right] r$$

$$= \frac{(n-1)}{n} \frac{\Gamma(n/2)}{\sqrt{(n-1)\pi(n-1)r/n}\,\Gamma[(n-1)/2]} \frac{1}{\{1+(n-1)r/[n(n-1)]\}^{n/2}}$$

$$= \frac{\Gamma(n/2)}{\sqrt{n\pi r}\,\Gamma[(n-1)/2]} \frac{1}{(1+r/n)^{n/2}} ,$$

which, to verify our method, we must obtain from equation (3.7.3).

Consider the path of integration Γ which we may express as follows:

$$\Gamma:\quad z = \frac{1}{2} + \frac{1}{2} e^{i\theta} .$$

Starting from $z = 0$, as z describes the circle in the clockwise direction θ ranges from $-\pi$, through an angle of 2π, to $+\pi$. Letting

$$(z-\beta) = R_\theta e^{i(\theta+2k\pi)}$$

we have

$$(z-\beta)^{-3/2} = R_\theta^{-3/2}\; e^{-i(3\theta/2+3k\pi)} \quad ,$$

which yields two solutions

$$(z-\beta)^{-3/2} = R_\theta^{-3/2}\; e^{-3\theta i/2} \qquad \text{(principle determination)}$$

and

$$(z-\beta)^{-3/2} = R_\theta^{-3/2}\; e^{-(3\theta/2+3\pi)i} \qquad \text{(secondary determination).}$$

To make $(z-\beta)^{-3/2}$ single-valued we take it to be the principle determination and cut the z-plane from $-\infty$ to β so that the secondary determination is unable to cross the cut. This procedure also makes $z^{(n-2)/2}$ single-valued, because in cutting the z-plane from $-\infty$ to β, it is automatically cut from $-\infty$ to 0.

Consider the closed path P consisting of the paths Γ', L_1, γ, and L_2 defined by

$$\Gamma': \quad z = \frac{1}{2} + \frac{1}{2}\, e^{i\theta} \quad , \pi - \varepsilon \geqslant \theta \geqslant -\pi + \varepsilon$$

$$L_1: \quad z = \beta + \tau e^{i(-\pi+\varepsilon)}, \; r(\varepsilon) \geqslant \tau \geqslant \delta$$

$$\gamma: \quad z = \beta + \delta e^{i\theta} \quad , \quad -\pi + \varepsilon \leqslant \theta \leqslant \pi - \varepsilon$$

$$L_2: \quad z = \beta + \tau e^{i(\pi-\varepsilon)} \quad , \quad \delta \leqslant \tau \leqslant r(\varepsilon) \; ,$$

where ε is an arbitrarily small angle, $r(\varepsilon)$ is the distance from $(\beta,0)$ to the point of intersection of L_1 and Γ, and δ is arbitrarily chosen such that $0 < \delta < \min(\beta,1-\beta)$. P is depicted in Figure 1.

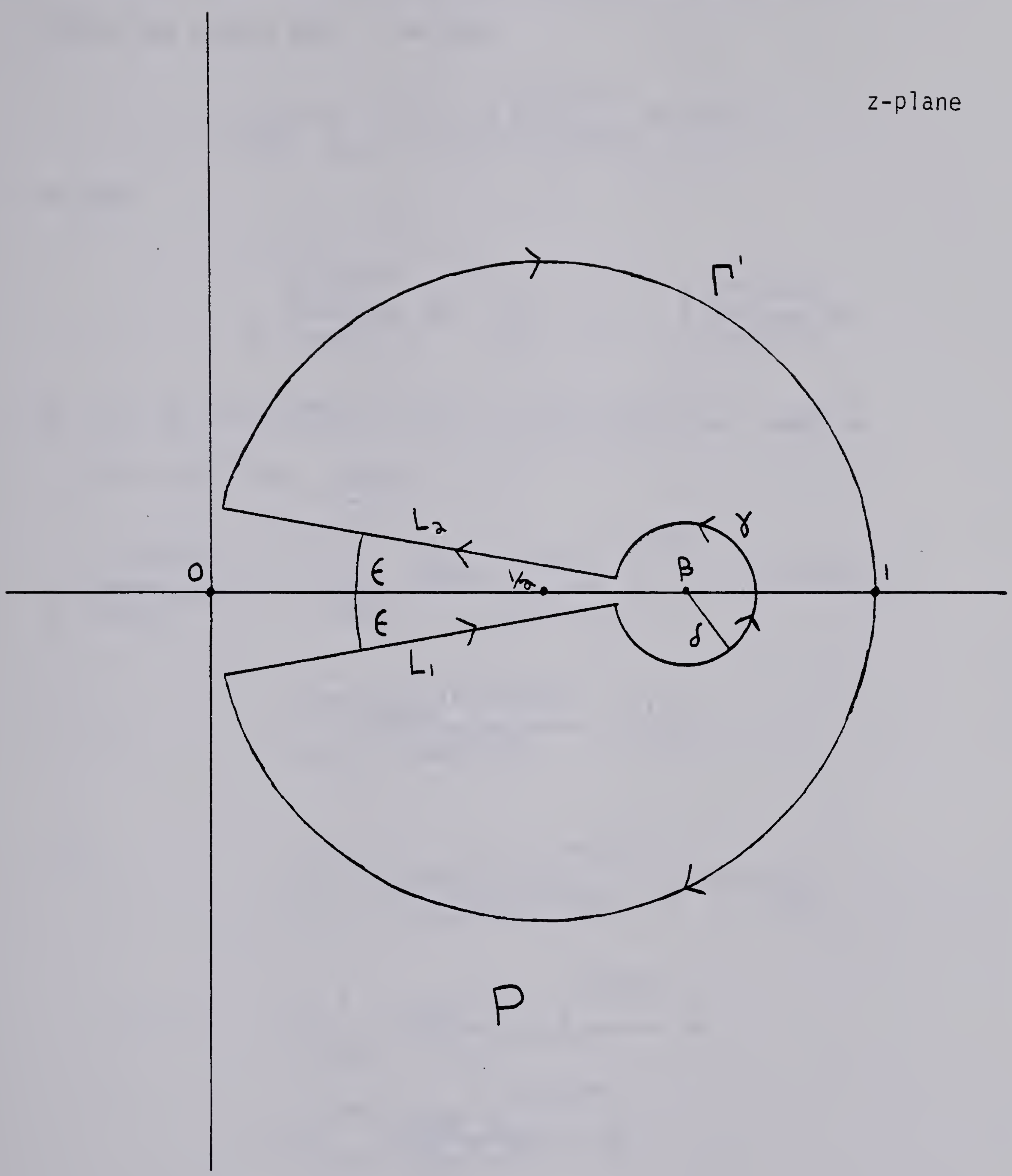

Figure 1: The path of integration P

Since $z^{(n-2)/2}(z-\beta)^{-3/2}$ is single-valued and analytic on and within the closed path P we have

$$\left\{\int_{\Gamma'} + \int_{L_1} + \int_{\gamma} + \int_{L_2}\right\} \frac{z^{(n-2)/2}}{(z-\beta)^{3/2}}\, dz = 0 \ ,$$

so that

$$\int_{\Gamma'} \frac{z^{(n-2)/2}}{(z-\beta)^{3/2}}\, dz = -\left\{\int_{L_1} + \int_{\gamma} + \int_{L_2}\right\} \frac{z^{(n-2)/2}}{(z-\beta)^{3/2}}\, dz \ .$$

On L_1 , $dz = e^{i(-\pi+\varepsilon)}\, d\tau$; on L_2, $dz = e^{i(\pi-\varepsilon)} d\tau$; and, on γ , $dz = \delta i e^{i\theta}\, d\theta$. Thus

$$\int_{\Gamma'} \frac{z^{(n-2)/2}}{(z-\beta)^{3/2}}\, dz = -\left\{\int_{r(\varepsilon)}^{\delta} \frac{(\beta+\tau e^{i(-\pi+\varepsilon)})^{(n-2)/2}}{(\tau e^{i(-\pi+\varepsilon)})^{3/2}}\, e^{i(-\pi+\varepsilon)}\, d\tau.\right.$$

$$+ \int_{-\pi+\varepsilon}^{\pi-\varepsilon} \frac{(\beta+\delta e^{i\theta})^{(n-2)/2}}{(\delta e^{i\theta})^{3/2}}\, \delta i e^{i\theta} d\theta$$

$$\left. + \int_{\delta}^{r(\varepsilon)} \frac{(\beta+\tau e^{i(\pi-\varepsilon)})^{(n-2)/2}}{(\tau\varepsilon^{i(\pi-\varepsilon)})^{3/2}}\, e^{i(\pi-\varepsilon)} d\tau \right\}$$

$$= -\left\{\int_{r(\varepsilon)}^{\delta} \frac{(\beta+\tau e^{i(-\pi+\varepsilon)})^{(n-2)/2}}{\tau^{3/2} e^{i(-\pi+\varepsilon)/2}}\, d\tau\right.$$

$$+ i\int_{-\pi+\varepsilon}^{\pi-\varepsilon} \frac{(\beta+\delta e^{i\theta})^{(n-2)/2}}{(\delta e^{i\theta})^{1/2}}\, d\theta$$

$$\left. + \int_{\delta}^{r(\varepsilon)} \frac{(\beta+\tau e^{i(\pi-\varepsilon)})^{(n-2)/2}}{\tau^{3/2} e^{i(\pi-\varepsilon)/2}}\, d\tau\right\}.$$

Letting $\varepsilon \to 0$ and $I = \int_{\Gamma} \frac{z^{(n-2)/2}}{(z-\beta)^{3/2}}\,dz$ we obtain

$$I = -\left\{\int_{\beta}^{\delta} \frac{(\beta+\tau e^{-i\pi})^{(n-2)/2}}{\tau^{3/2}e^{-i\pi/2}}\,d\tau + i\int_{-\pi}^{\pi} \frac{(\beta+\delta e^{i\theta})^{(n-2)/2}}{(\delta e^{i\theta})^{1/2}}\,d\theta\right.$$

$$\left. + \int_{\delta}^{\beta} \frac{(\beta+\tau e^{i\pi})^{(n-2)/2}}{\tau^{3/2}e^{i\pi/2}}\,d\tau\right\}$$

$$= -\left\{i\int_{\beta}^{\delta} \frac{(\beta-\tau)^{(n-2)/2}}{\tau^{3/2}}\,d\tau + \frac{i}{\sqrt{\delta}}\int_{-\pi}^{\pi} \frac{(\beta+\delta e^{i\theta})^{(n-2)/2}}{(e^{i\theta})^{1/2}}\,d\theta\right.$$

$$\left. - i\int_{\delta}^{\beta} \frac{(\beta-\tau)^{(n-2)/2}}{\tau^{3/2}}\,d\tau\right\}$$

or

(3.7.5) $$I = i\left\{2I_1 - \frac{1}{\sqrt{\delta}}\,I_2\right\} \quad ,$$

where

$$I_1 = \int_{\delta}^{\beta} \frac{(\beta-\tau)^{(n-2)/2}}{\tau^{3/2}}\,d\tau \quad \text{and} \quad I_2 = \int_{-\pi}^{\pi} \frac{(\beta+\delta e^{i\theta})^{(n-2)/2}}{e^{i\theta/2}}\,d\theta \ .$$

Consider

$$I_1 = \beta^{(n-2)/2}\int_{\delta}^{\beta}\left(1 - \frac{\tau}{\beta}\right)^{(n-2)/2}\tau^{-3/2}\,d\tau \ .$$

Let $u = \tau/\beta$. Then $\tau = \beta u$ and $d\tau = \beta\,du$, giving

$$I_1 = \beta^{(n-2)/2}\int_{\delta/\beta}^{1} (1-u)^{(n-2)/2}(\beta u)^{-3/2}\beta\,du$$

$$= \beta^{(n-3)/2} \int_{\delta/\beta}^{1} (1-u)^{(n-2)/2} u^{-3/2} du .$$

Applying the general binomial formula, we see that

$$\sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(n/2)}{\Gamma(k+1)\Gamma(n/2-k)} u^k$$

converges uniformly to $(1-u)^{(n-2)/2}$ for $0 \leqslant u < 1$. Thus

$$I_1 = \beta^{(n-3)/2} \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(n/2)}{\Gamma(k+1)\Gamma(n/2-k)} \int_{\delta/\beta}^{1} u^{k-3/2} du$$

$$= \beta^{(n-3)/2} \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(n/2)}{\Gamma(k+1)\Gamma(n/2-k)} \left[\frac{u^{k-1/2}}{(k-1/2)}\right]_{\delta/\beta}^{1}$$

and so

(3.7.6) $$2I_1 = 2\beta^{(n-3)/2}\Gamma(n/2) \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+1)\Gamma(n/2-k)(k-1/2)}$$

$$- \frac{2\beta^{(n-2)/2}}{\sqrt{\delta}} \Gamma(n/2) \sum_{k=0}^{\infty} \frac{(-1)^k(\delta/\beta)^k}{\Gamma(k+1)\Gamma(n/2-k)(k-1/2)} .$$

Now consider

$$I_2 = \beta^{(n-2)/2} \int_{-\pi}^{\pi} \left(1 + \frac{\delta}{\beta} e^{i\theta}\right)^{(n-2)/2} e^{-i\theta/2} d\theta .$$

Since $|\delta e^{i\theta}/\beta| < 1$, we have, by the general binomial formula, that

$$\sum_{k=0}^{\infty} \frac{\Gamma(n/2)}{\Gamma(k+1)\Gamma(n/2-k)} \left(\frac{\delta}{\beta} e^{i\theta}\right)^k$$

converges to $\left(1 + \frac{\delta}{\beta} e^{i\theta}\right)^{(n-2)/2}$ uniformly in θ . Therefore

$$I_2 = \beta^{(n-2)/2}\Gamma(n/2) \sum_{k=0}^{\infty} \frac{(\delta/\beta)^k}{\Gamma(k+1)\Gamma(n/2-k)} \int_{-\pi}^{\pi} e^{i\theta(k-1/2)}\, d\theta$$

$$= \beta^{(n-2)/2}\Gamma(n/2) \sum_{k=0}^{\infty} \frac{(\delta/\beta)^k}{\Gamma(k+1)\Gamma(n/2-k)} \left[\frac{e^{i\theta(k-1/2)}}{i(k-1/2)}\right]_{-\pi}^{\pi}$$

$$= \beta^{(n-2)/2}\Gamma(n/2) \sum_{k=0}^{\infty} \frac{(\delta/\beta)^k}{\Gamma(k+1)\Gamma(n/2-k)} \frac{2}{(k-1/2)} \sin[\pi(k-1/2)] .$$

Noting that $\sin[\pi(k-1/2)] = (-1)^{k+1}$, we obtain

$$\text{(3.7.7)} \qquad \frac{-1}{\sqrt{\delta}} I_2 = \frac{2\beta^{(n-2)/2}}{\sqrt{\delta}} \Gamma(n/2) \sum_{k=0}^{\infty} \frac{(-1)^k(\delta/\beta)^k}{\Gamma(k+1)\Gamma(n/2-k)(k-1/2)} .$$

Combining equations (3.7.5), (3.7.6) and (3.7.7) we have

$$I = \frac{2i\beta^{(n-2)/2}}{\sqrt{\delta}} \Gamma(n/2)\left[\sum_{k=0}^{\infty} \frac{(-1)^k(\delta/\beta)^k}{\Gamma(k+1)\Gamma(n/2-k)(k-1/2)}\right.$$

$$\left. - \sum_{k=0}^{\infty} \frac{(-1)^k(\delta/\beta)^k}{\Gamma(k+1)\Gamma(n/2-k)(k-1/2)}\right]$$

$$+ 2i\beta^{(n-3)/2}\Gamma(n/2) \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+1)\Gamma(n/2-k)(k-1/2)}$$

$$= 2i \; \frac{n}{n+r} \quad {}^{(n-3)/2} \quad K_n$$

where $K_n = \Gamma(n/2) \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+1)\Gamma(n/2-k)(k-1/2)}$. From equation (3.7.3) we obtain

$$h(r) = -\frac{1}{4\pi i\sqrt{nr}(1+r/n)^{3/2}}\, I$$

$$= -\frac{1}{2\pi\sqrt{nr}(1+r/n)^{3/2}}\left(\frac{n+r}{n}\right)^{-(n-3)/2} K_n \ ,$$

and thus

$$h(r) = -\frac{1}{\sqrt{\pi rn}\ (1+r/n)^{n/2}}\ \frac{K_n}{2\sqrt{\pi}} \ . \tag{3.7.8}$$

Comparing equations (3.7.4) and (3.7.8), we see that it remains to show that

$$\frac{K_n}{2\sqrt{\pi}} = -\frac{\Gamma(n/2)}{\Gamma[(n-1)/2]} \ .$$

Renormalization yields this result since $h(r)$ is a probability density function.

3.8 An Asymptotic Approximation for h(r).

In this section we obtain an asymptotic approximation for $h(r)$ for the general case $0 < |\rho| < 1$. From equations (3.5.4), (3.5.6) and (3.6.6) we have

$$|B| \approx \frac{u^{n-1}(u+f-a)}{(u-v)(2e+d)^2}\{(u+f-a)[(2e+d)^2-c((u-v)-n(2e+d))] + c(2e+d)(u-f+a)\}$$

where $u = z^{-1}$, $f - a = -\rho^2$, $v = \rho^2 z$, $e = -\rho$, and $d = z^{-1} + \rho^2 z$. It follows that

$$u + f - a = z^{-1} - \rho^2 = (1-\rho^2 z)z^{-1} \quad ,$$

$$u - f + a = z^{-1} + \rho^2 = (1+\rho^2 z)z^{-1} \quad ,$$

$$u - v = z^{-1} - \rho^2 z = (1-\rho z)(1+\rho z)z^{-1} \quad ,$$

and

$$2e + d = -2\rho + z^{-1} + \rho^2 z = (1-\rho z)^2 z^{-1} \; .$$

Thus

$$\frac{u^{n-1}(u+f-a)}{(u-v)(2e+d)^2} = \frac{z^{-(n-1)}(1-\rho^2 z)z^{-1}}{(1-\rho z)(1+\rho z)z^{-1}[(1-\rho z)^2 z^{-1}]^2} = \frac{(1-\rho^2 z)}{z^{n-3}(1-\rho z)^5(1+\rho z)}$$

and

$$|B| \approx \frac{(1-\rho^2 z)}{z^{n-3}(1-\rho z)^5(1+\rho z)}\left\{\frac{(1-\rho^2 z)}{z}\left[\frac{(1-\rho z)^4}{z^2} - c\left(\frac{(1-\rho z)(1+\rho z)}{z} - n\,\frac{(1-\rho z)^2}{z}\right)\right]\right.$$

$$\left. + \, c\,\frac{(1-\rho z)^2}{z}\,\frac{(1+\rho z)^2}{z}\right\}$$

$$= \frac{(1-\rho^2 z)}{z^{n-1}(1-\rho z)^4(1+\rho z)}\Big\{(1-\rho^2 z)\Big[\frac{(1-\rho z)^3}{z} - c(1+\rho z-n(1-\rho z))\Big]$$

$$+ c(1-\rho z)(1+\rho^2 z)\Big\}$$

By equation (3.5.5), we have

$$M(T_0,T_1) = \frac{(1-\rho^2)^{1/2}}{|\underline{B}|^{1/2}}$$

$$\approx (1-\rho^2)^{1/2}\Big[\frac{(1-\rho^2 z)}{z^{n-1}(1-\rho z)^4(1+\rho z)}\Big]\Big\{(1-\rho^2 z)\Big[\frac{(1-\rho z)^3}{z} - c(1+\rho z-n+n\rho z)\Big]$$

$$+ c(1-\rho z)(1+\rho^2 z)\Big\}\Big]^{-1/2} .$$

From Equation (3.6.2) we obtain

$$(3.8.1)\quad T_1 = \frac{1}{2nr}\,[z^{-1} + \rho^2 z - 1 - \rho^2 + 2n\mu] = \frac{1}{2nr}\,\Big[\frac{(1-z)(1-\rho^2 z)}{2} + 2n\mu\Big]$$

so that

$$\frac{\partial T_1}{\partial z} = \frac{1}{2nr}\,(-z^{-2}+\rho^2) = -\,\frac{(1-\rho^2 z^2)}{2nrz^2}$$

and

$$\frac{\partial}{\partial\mu}\, M(T_0,T_1)\left[\frac{\partial T_1}{\partial z}\right]\Bigg|_{\mu=0}$$

$$\approx -\frac{(1-\rho^2z^2)(1-\rho^2)^{1/2}}{2nrz^2}\left[\frac{(1-\rho^2 z)}{z^{n-1}(1-\rho z)^4(1+\rho z)}\left\{(1-\rho^2 z)\left[\frac{(1-\rho z)^3}{z} - c(1+\rho z-n+n\rho z)\right]\right.\right.$$

$$\left.\left.+ c(1-\rho z)(1+\rho^2 z)\right\}\right]^{-3/2}(-1/2)$$

$$\cdot\left[\frac{(1-\rho^2 z)}{z^{n-1}(1-\rho z)^4(1+\rho z)}\left(-(1-\rho^2 z)(1+\rho z-n+n\rho z) + (1-\rho z)(1+\rho^2 z)\right)\right]\frac{\partial c}{\partial\mu}\bigg|_{\mu=0} .$$

Combining equations (3.5.4) and (3.8.1) we find that

$$\begin{aligned} c &= 2T_0 - 2nT_1 \\ &= 2\mu - 2(n+r)T_1 \\ &= 2\mu - \frac{2(n+r)}{2nr}\left[\frac{(1-z)(1-\rho^2 z)}{z} + 2n\mu\right] \\ &= -\frac{(n+r)}{nr}\,\frac{(1-z)(1-\rho^2 z)}{z} - \frac{2n\mu}{r} . \end{aligned}$$

Therefore

$$c\bigg|_{\mu=0} = -\frac{(n+r)}{nr}\,\frac{(1-z)(1-\rho^2 z)}{z} ,$$

$$\frac{\partial c}{\partial\mu}\bigg|_{\mu=0} = -\frac{2n}{r} ,$$

and

$$(3.8.2)\quad \frac{\partial}{\partial\mu} M(T_0,T_1)\left[\frac{\partial T_1}{\partial z}\right]\Bigg|_{\mu=0}$$

$$\approx \frac{(1-\rho^2z^2)(1-\rho^2)^{1/2}}{4nrz^2}\left[\frac{(1-\rho^2z)}{z^{n-1}(1-\rho z)^4(1+\rho z)}\right.$$

$$\cdot\left\{(1-\rho^2z)\left[\frac{(1-\rho z)^3}{z} + \frac{(n+r)}{nr}\,\frac{(1-z)(1-\rho^2z)}{z}\,(1+\rho z-n+n\rho z)\right]\right.$$

$$\left.\left. - \frac{(n+r)}{nr}\,\frac{(1-z)(1-\rho^2z)}{z}\,(1-\rho z)(1+\rho^2z)\right\}\right]^{-3/2}$$

$$\cdot\left[\frac{(1-\rho^2z)}{z^{n-1}(1-\rho z)^4(1+\rho z)}\left(-\ (1-\rho^2z)(1+\rho z-n+n\rho z)+(1-\rho z)(1+\rho^2z)\right)\right]\left(\frac{-2n}{r}\right)$$

$$= -\ \frac{(1-\rho^2z^2)(1-\rho^2)^{1/2}(1-\rho^2z)}{2r^2z^{n+1}(1-\rho z)^4(1+\rho z)}\,[-(1-\rho^2z)(1+\rho z-n+n\rho z) + (1-\rho z)(1+\rho^2z)]Q\ ,$$

where

$$Q = \left[\frac{(1-\rho^2z)}{z^{n-1}(1-\rho z)^4(1+\rho z)}\left\{(1-\rho^2z)\left[\frac{(1-\rho z)^3}{z} + \frac{(n+r)}{nr}\,\frac{(1-z)(1-\rho^2z)}{z}\,(1+\rho z-n+n\rho z)\right]\right.\right.$$

$$\left.\left. - \frac{(n+r)}{nr}\,\frac{(1-z)(1-\rho^2z)}{z}\,(1-\rho z)(1+\rho^2z)\right\}\right]^{-3/2} .$$

Consider

$$Q = \left[\frac{(1-\rho^2z)^2}{z^n(1-\rho z)^4(1+\rho z)}\left\{(1-\rho z)^3 + \frac{(n+r)}{nr}(1-z)(1-\rho^2z)(1+\rho z-n-n\rho z)\right.\right.$$

$$\left.\left. - \frac{(n+r)}{nr}(1-z)(1-\rho z)(1+\rho^2z)\right\}\right]^{-3/2}$$

$$= \left[\frac{(1-\rho^2z)^2}{z^n(1-\rho z)^4(1+\rho z)nr}\left\{nr(1-\rho z)^3 + (n+r)(1-z)[(1-\rho^2z)(1+\rho z)\right.\right.$$

$$\left.\left. - n(1-\rho^2z)(1-\rho z) - (1-\rho z)(1+\rho^2z)]\right\}\right]^{-3/2} .$$

Observing that $(1-\rho^2z)(1+\rho z) - (1-\rho z)(1+\rho^2z) = 2\rho z(1-\rho)$, we obtain

$$Q = \left[\frac{(1-\rho^2z)^2}{z^n(1-\rho z)^4(1+\rho z)nr}\left\{nr(1-\rho z)^3\right.\right.$$

$$\left.\left. + (n+r)(1-z)[2\rho z(1-\rho) - n(1-\rho^2z)(1-\rho z)]\right\}\right]^{-3/2} .$$

Neglecting $2\rho z(1-\rho)$ in comparison with $n(1-\rho^2z)(1-\rho z)$ we incur an error which is relatively $O(n^{-1})$, and thus

$$Q \approx \left[\frac{(1-\rho^2z)^2}{z^n(1-\rho z)^4(1+\rho z)nr}\left\{nr(1-\rho z)^3 - (n+r)(1-z)n(1-\rho^2z)(1-\rho z)\right\}\right]^{-3/2}$$

$$= \left[\frac{(1-\rho^2z)^2}{z^n(1-\rho z)^3(1+\rho z)r}\, L\right]^{-3/2} ,$$

where

$$L = r(1-\rho z)^2 - (n+r)(1-z)(1-\rho^2 z)$$

$$= [r - (n+r)] + z[-2r\rho + (n+r)(1+\rho^2)] + z^2\rho^2[r - (n+r)]$$

$$= -n + z[n(1+\rho^2) + r(1-\rho)^2] - n\rho^2 z^2 \; .$$

Letting β_1 and β_2 denote the roots of $L = 0$, we may express L as $L = - n\rho^2(z-\beta_1)(z-\beta_2)$, so that

$$Q \approx \left[\frac{-n\rho^2(1-\rho^2 z)^2(z-\beta_1)(z-\beta_2)}{z^n(1-\rho z)^3(1+\rho z)r}\right]^{-3/2} \tag{3.8.3}$$

where $0 < |\rho| < 1$,

$$\beta_1 = \frac{n(1+\rho^2) + r(1-\rho^2) - \sqrt{[n(1+\rho^2) + r(1-\rho)^2]^2 - 4n^2\rho^2}}{2n\rho^2}$$

and

$$\beta_2 = \frac{n(1+\rho^2) + r(1-\rho)^2 + \sqrt{[n(1+\rho^2) + r(1-\rho)^2]^2 - 4n^2\rho^2}}{2n\rho^2} \; .$$

Let us digress briefly to examine the roots β_1 and β_2 . The expression under the radical sign,

$$[n(1+\rho^2) + r(1-\rho)^2]^2 - 4n^2\rho^2$$

$$= n^2(1+\rho^2)^2 + 2nr(1+\rho^2)(1-\rho)^2 + r^2(1-\rho)^4 - 4n^2\rho^2$$

$$= n^2(1-\rho^2)^2 + 2nr(1+\rho^2)(1-\rho)^2 + r^2(1-\rho)^4 \; ,$$

is real and positive. Thus both β_1 and β_2 are real and $\beta_2 > 0$. Furthermore,

$$\beta_1\beta_2 = \frac{[n(1+\rho^2) + r(1-\rho)^2]^2 - \{[n(1+\rho^2) + r(1-\rho)^2]^2 - 4n^2\rho^2\}}{4n^2\rho^4}$$

$$= \frac{4n^2\rho^2}{4n^2\rho^4} = \frac{1}{\rho^2} ,$$

and so, $\beta_1 > 0$. Also, since

$$n^2(1-\rho^2)^2 + 2nr(1+\rho^2)(1-\rho)^2 + r^2(1-\rho)^4$$

$$= n^2(1-\rho^2)^2 + 2nr(1-\rho^2)(1-\rho)^2 + r^2(1-\rho)^4 + 4nr\rho^2(1-\rho)^2$$

$$> n^2(1-\rho^2)^2 + 2nr(1-\rho^2)(1-\rho)^2 + r^2(1-\rho)^4$$

$$= [n(1-\rho^2) + r(1-\rho)^2]^2$$

we have that

$$\beta_1 = \frac{n(1+\rho^2) + r(1-\rho)^2 - \sqrt{n^2(1-\rho^2)^2+2nr(1+\rho^2)(1-\rho)^2+r^2(1-\rho)^4}}{2n\rho^2}$$

$$< \frac{n(1+\rho^2) + r(1-\rho)^2 - \sqrt{[n(1-\rho^2) + r(1-\rho)^2]^2}}{2n\rho^2}$$

$$= \frac{2n\rho^2}{2n\rho^2} = 1 .$$

Thus $0 < \beta_1 < 1$ and $\beta_2 = 1/(\rho^2\beta_1) > 1/\rho^2 > 1$.

From equations (3.8.2) and (3.8.3), we obtain

$$\frac{\partial}{\partial\mu} M(T_0,T_1)\left[\frac{\partial T_1}{\partial z}\right]\Bigg|_{\mu=0}$$

$$\approx -\frac{(1-\rho^2z^2)(1-\rho^2)^{1/2}(1-\rho^2z)}{2r^2z^{n+1}(1-\rho z)^4(1+\rho z)}[-(1-\rho^2z)(1+\rho z-n+n\rho z) + (1-\rho z)(1+\rho^2z)]$$

$$\cdot\left[\frac{-n\rho^2(1-\rho^2z)^2(z-\beta_1)(z-\beta_2)}{z^n(1-\rho z)^3(1+\rho z)r}\right]^{-3/2}$$

$$= \frac{(1-\rho^2z^2)(1-\rho^2)^{1/2}(1-\rho^2z)}{2r^2z^{n+1}(1-\rho z)^4(1+\rho z)}[(1-\rho^2z)(1+\rho z) - n(1-\rho^2z)(1-\rho z) - (1-\rho z)(1+\rho^2z)]$$

$$\cdot\left[\frac{-n\rho^2(1-\rho^2z)^2(z-\beta_1)(z-\beta_2)}{z^n(1-\rho z)^3(1+\rho z)r}\right]^{-3/2} .$$

Once again, we oberve that by neglecting the terms $(1-\rho^2)(1+\rho z) - (1-\rho z)(1+\rho^2z)$ in comparison with $n(1-\rho^2z)(1-\rho z)$ we introduce an error which is relative $O(n^{-1})$. Thus

$$\frac{\partial}{\partial\mu} M(T_0,T_1)\left[\frac{\partial T_1}{\partial z}\right]\Bigg|_{\mu=0}$$

$$\approx \frac{(1-\rho^2z^2)(1-\rho^2)^{1/2}(1-\rho^2z)}{2r^2z^{n+1}(1-\rho z)^4(1+\rho z)}[-n(1-\rho^2z)(1-\rho)]$$

$$\cdot\left[\frac{-z^n(1-\rho z)^3(1+\rho z)r}{n\rho^2(1-\rho^2z)^2(z-\beta_1)(z-\beta_2)}\right]^{3/2} ,$$

which upon simplification becomes

$$(3.8.4) \qquad \frac{\partial}{\partial \mu} M(T_0,T_1)\left[\frac{\partial T_1}{\partial z}\right]\Bigg|_{\mu=0} \approx \frac{i(1-\rho^2)^{1/2} z^{(n-2)/2}(1-\rho z)^{5/2}(1+\rho z)^{3/2}}{2\rho\sqrt{nr}(1-\rho^2 z)(z-\beta_1)^{3/2}(z-\beta_2)^{3/2}} .$$

It follows that

$$(3.8.5) \qquad h(r) = \frac{1}{2\pi i}\int_{\Gamma^*} \frac{\partial}{\partial u}\left[M(\mu - rT_1, T_1)\ \frac{\partial T_1}{\partial z}\right]\Bigg|_{\mu=0} dz$$

$$\approx \frac{1}{2\pi i}\int_{\Gamma^*} \frac{i(1-\rho^2)^{1/2} z^{(n-2)/2}(1-\rho z)^{5/2}(1+\rho z)^{3/2}}{2\rho^3\sqrt{nr}\ (1-\rho^2 z)(z-\beta_1)^{3/2}(z-\beta_2)^{3/2}}\, dz$$

$$= \frac{(1-\rho^2)^{1/2}}{4\pi\rho^3\sqrt{nr}} \int_{\Gamma^*} \frac{f(z)}{(z-\beta_1)^{3/2}}\, dz$$

where

$$f(z) = \frac{z^{(n-2)/2}(1-\rho z)^{5/2}(1+\rho z)^{3/2}}{(1-\rho^2 z)(z-\beta_2)^{3/2}} ,$$

$z = x + iy$, $|z| < 1$, and Γ^* is the path of integration

$$\Gamma^* : |y| = \sqrt{\frac{x(1-x)(1-\rho^2 x)}{1+\rho^2(1-x)}} .$$

From the analysis of the path of integration presented in section 3.6, we know: (i) Γ^* is a closed path which is symmetric about $y = 0$; (ii) $0 \leqslant x \leqslant 1$; (iii) $0 \leqslant |y| \leqslant 1/2$; (iv) as T_1 travels along the imaginary axis from $-i\infty$ to 0, z moves in a clockwise direction in

quadrant I from (0,0) to (1,0) ; and (v) as T_1 goes along the imaginary axis from 0 to $+i\infty$, z proceeds clockwise in quadrant IV form (1,0) to (0,0). The path Γ^* is shown in Figure 2.

Consider the integrand

$$\frac{f(z)}{(z-\beta_1)^{3/2}} = \frac{z^{(n-2)/2}(1-\rho z)^{5/2}(1+\rho z)^{3/2}}{(1-\rho^2 z)(z-\beta_1)^{3/2}(z-\beta_2)^{3/2}}$$

of h(r). There are branch points at $z = \beta_1$, β_2 , $1/\rho$, $-1/\rho$, and, if n is odd, at $z = 0$. Letting

$$(z-\beta_1) = R(\theta,\beta_1)e^{i(\theta+2k\pi)}$$

we have

$$(z-\beta_1)^{-3/2} = [R(\theta,\beta_1)]^{-3/2}e^{-i(3\theta/2+3k\pi)} \quad ,$$

which gives two solutions

$$(z-\beta_1)^{-3/2} = [(R(\theta,\beta_1)]^{-3/2}e^{-i3\theta/2} \qquad \text{(principle determination)}$$

and

$$(z-\beta_1)^{-3/2} = [R(\theta,\beta_1)]^{-3/2}e^{-(3\theta/2+3\pi)i} \qquad \text{(secondary determination).}$$

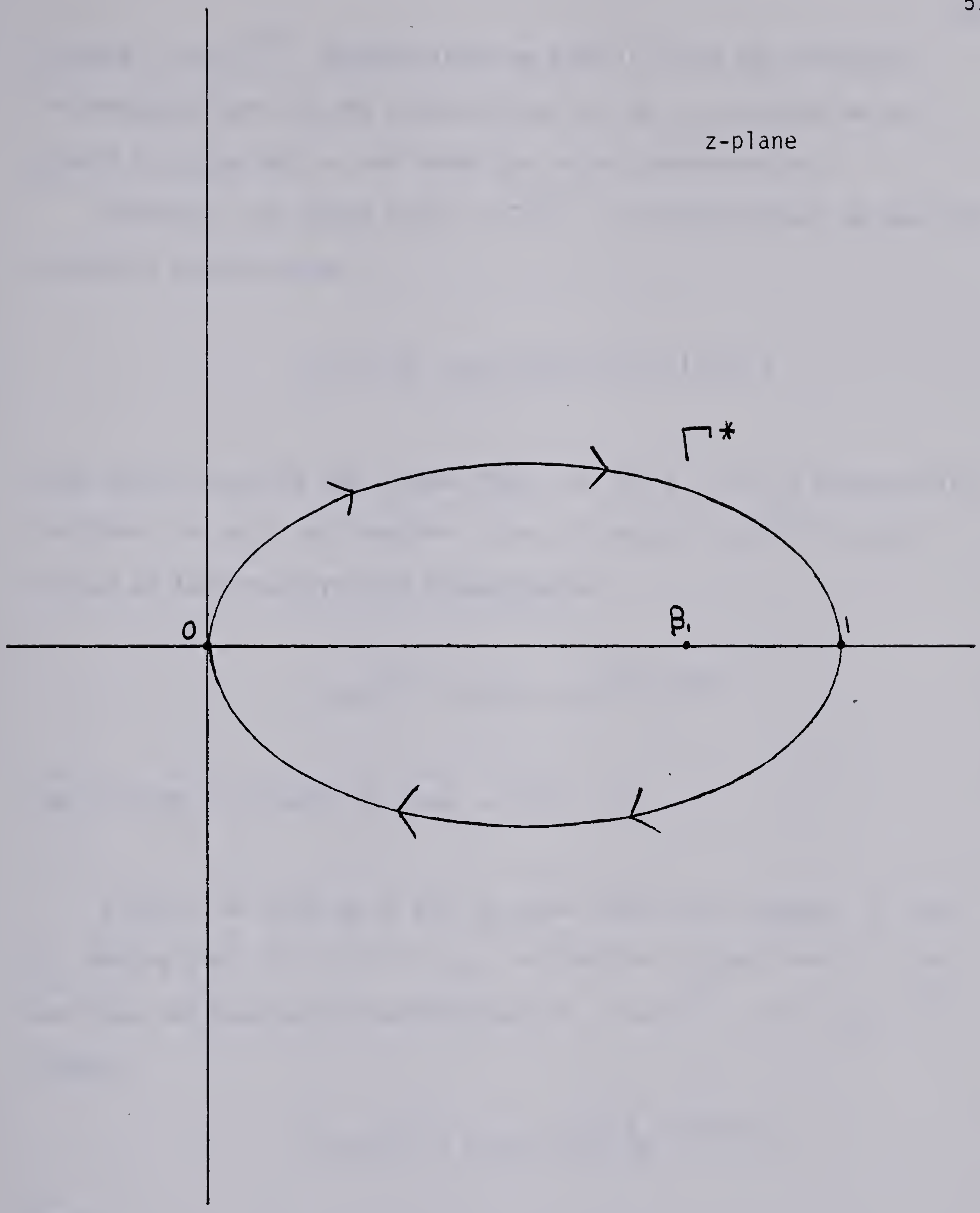

Figure 2: The path of integration Γ^*

To make $(z-\beta_1)^{-3/2}$ single-valued we take it to be the principle determination and cut the z-plane from $-\infty$ to β_1 so that we are unable to cross the cut and reach the second determination.

Similarly, to ensure that $z^{(n-2)/2}$ is single-valued, we take the principle determination

$$z^{(n-2)/2} = [R(\theta,0)]^{(n-2)/2} e^{i\theta(n-2)/2} .$$

Note that in cutting the z-plane from $-\infty$ to β_1, it is automatically cut from $-\infty$ to 0, as required. Also, to make $(1+\rho z)^{3/2}$ single-valued we take the principle determination

$$(1+\rho z)^{3/2} = [R[\theta,-1/\rho)]^{3/2} e^{i3\theta/2} \;;$$

the z-plane is already cut from $-\infty$ to $-1/\rho$.

Finally, we wish to be able to cross the x-axis between β_1 and 1. Noting that $1/\rho \leqslant 1/\rho^2 < \beta_2$, we cut the z-plane from $1/\rho$ to ∞ and take the secondary determinations of $(1-\rho z)^{5/2}$ and $(z-\beta_2)^{-3/2}$. Taking

$$(1-\rho z)^{5/2} = [R(\theta,1/\rho)]^{5/2} e^{i(5\theta/2+5\pi)}$$

and

$$(z-\beta_2)^{-3/2} = [R(\theta,\beta_2)]^{-3/2} e^{-i(3\theta/2+3\pi)} ,$$

we make $(1-\rho z)^{5/2}$ and $(z-\beta_2)^{-3/2}$ single-valued, and avoid the

singularity of the integrand at $z = 1/\rho^2$.

Consider the closed path P^* consisting of the paths $\Gamma^{*\prime}$, L_1^*, γ^*, and L_2^* defined by

$$\Gamma^{*\prime}: \quad \Gamma^* , \quad \pi - \varepsilon \geqslant \theta \geqslant -\pi + \varepsilon$$

$$L_1^*: \; z = \beta_1 + \tau e^{i(-\pi+\varepsilon)} \quad , \quad r(\varepsilon) \geqslant \tau \geqslant \delta$$

$$\gamma^*: \; z = \beta_1 + \delta e^{i\theta} \quad , \quad -\pi + \varepsilon \leqslant \theta \leqslant \pi - \varepsilon$$

$$L_2^*: \; z = \beta_1 + \tau^{i(\pi-\varepsilon)} \quad , \quad \delta \leqslant \tau \leqslant r(\varepsilon)$$

where ε is an arbitrarily small angle, $r(\varepsilon)$ is the distance from $(\beta_1,0)$ to the point at which L_1^* and Γ^* intersect, and δ is arbitrarily chosen such that $0 < \delta < \min(\beta_1, 1-\beta_1)$. P^* is illustrated in Figure 3. On and within P^*, $f(z)(z-\beta_1)^{-3/2}$ is single-valued and analytic. Thus

$$\left\{ \int_{\Gamma^{*\prime}} + \int_{L_1^*} + \int_{\gamma^*} + \int_{L_2^*} \right\} \frac{f(z)}{(z-\beta_1)^{3/2}} \, dz = 0$$

or

$$\int_{\Gamma^{*\prime}} \frac{f(z)}{(z-\beta_1)^{3/2}} \, dz = - \left\{ \int_{L_1^*} + \int_{\gamma^*} + \int_{L_2^*} \right\} \frac{f(z)}{(z-\beta_1)^{3/2}} \, dz \; . \tag{3.8.6}$$

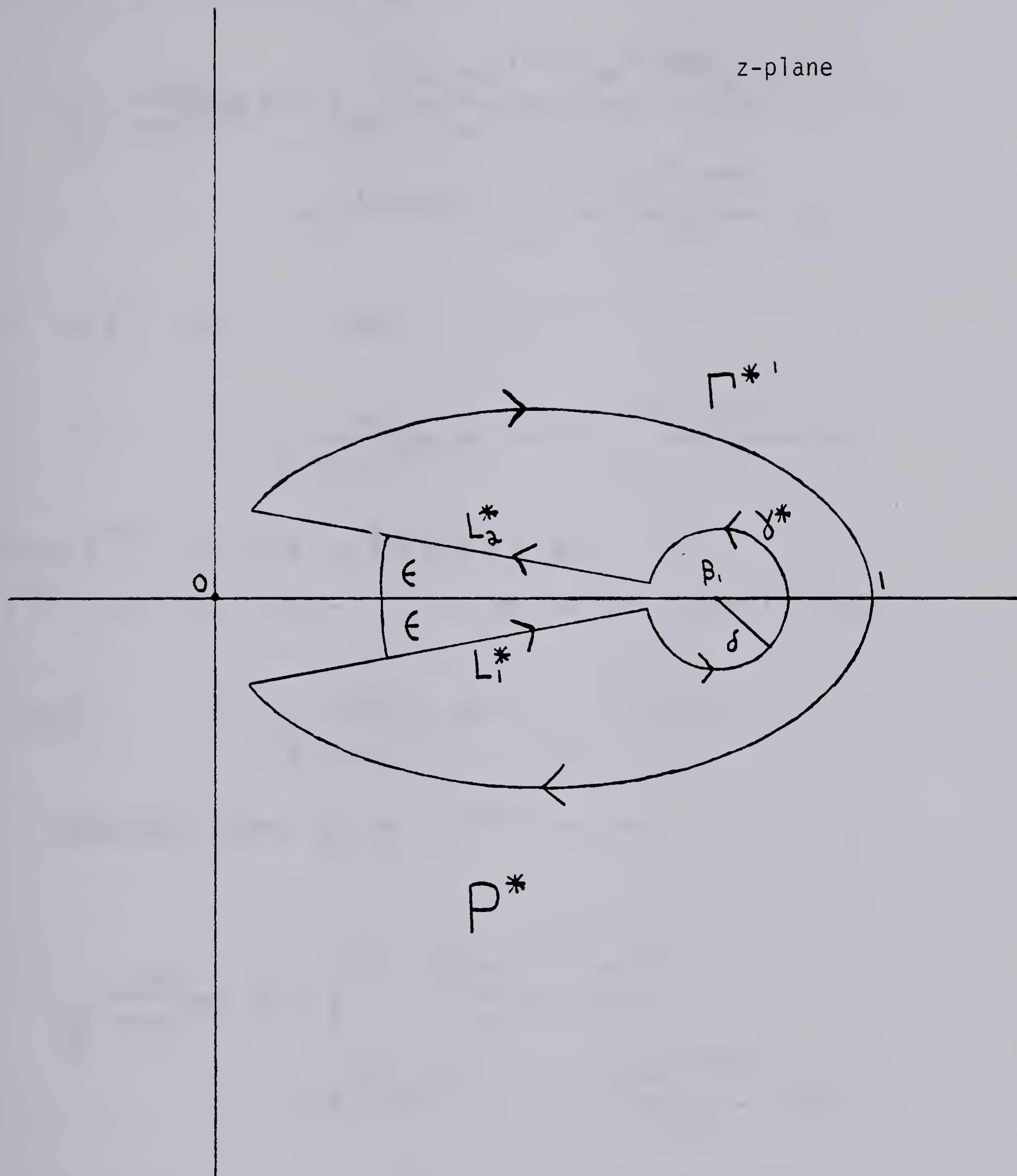

Figure 3: The path of integration P*

On L_1^*, $dz = e^{i(-\pi+\varepsilon)}d\tau$ so that

$$\int_{L_1^*} \frac{f(z)}{(z-\beta_1)^{3/2}}\, dz = \int_{r(\varepsilon)}^{\delta} \frac{f(\beta_1+\tau e^{i(-\pi+\varepsilon)})e^{i(-\pi+e)}}{(\tau e^{i(-\pi+\varepsilon)})^{3/2}}\, d\tau$$

$$= e^{-i(-\pi+\varepsilon)/2} \int_{r(\varepsilon)}^{\delta} \frac{f(\beta_1+\tau e^{i(-\pi+\varepsilon)})}{\tau^{3/2}}\, d\tau \ .$$

As $\varepsilon \to 0$, $r(\varepsilon) \to \beta_1$, and

$$\int_{L_1^*} \frac{f(z)}{(z-\beta_1)^{3/2}}\, dz \to e^{i\pi/2} \int_{\beta_1}^{\delta} \frac{f(\beta_1+\tau e^{-i\pi})}{\tau^{3/2}}\, d\tau \ .$$

Since $e^{i\pi/2} = \cos \pi/2 + i \sin \pi/2 = i$ and $e^{-i\pi} = \cos(-\pi) + i \sin(-\pi) = -1$, we see that in the limit as $\varepsilon \to 0$

(3.8.7) $$\int_{L_1^*} \frac{f(z)}{(z-\beta_1)^{3/2}}\, dz = i \int_{\beta_1}^{\delta} \frac{f(\beta_1-\tau)}{\tau^{3/2}}\, d\tau \ .$$

Similarly, along L_2^*, $dz = e^{i(\pi-\varepsilon)}\, d\tau$ and

$$\int_{L_2^*} \frac{f(z)}{(z-\beta_1)^{3/2}}\, dz = \int_{\delta}^{r(\varepsilon)} \frac{f(\beta_1+\tau e^{i(\pi-\varepsilon)})e^{i(\pi-\varepsilon)}}{(\tau e^{i(\pi-\varepsilon)})^{3/2}}\, d\tau$$

$$= e^{-i(\pi-\varepsilon)/2} \int_{\delta}^{r(\varepsilon)} \frac{f(\beta_1+\tau e^{i(\pi-\varepsilon)})}{\tau^{3/2}}\, d\tau \ .$$

Observing that $e^{-i\pi/2} = \cos(-\pi/2) + i \sin(-\pi/2) = -i$ and $e^{i\pi} = \cos(\pi) + i \sin \pi = -1$, we obtain, in the limit as $\varepsilon \to 0$,

(3.8.8) $$\int_{L_2^*} \frac{f(z)}{(z-\beta_1)^{3/2}}\, dz = -\, i \int_{\delta}^{\beta_1} \frac{f(\beta_1-\tau)}{\tau^{3/2}}\, d\tau \ .$$

On γ^* , $dz = i\delta e^{i\theta}d\theta$ and so

$$\int_{\gamma^*} \frac{f(z)}{(z-\beta_1)^{3/2}}\, dz = \int_{-\pi+\varepsilon}^{\pi-\varepsilon} \frac{f(\beta_1+\delta e^{i\theta})\delta e^{i\theta} i}{(\delta e^{i\theta})^{3/2}}\, d\theta \ ,$$

which, in the limit as $\varepsilon \to 0$ becomes

$$H = \frac{i}{\sqrt{\delta}} \int_{-\pi}^{\pi} f(\beta_1+\delta e^{i\theta}) e^{-i\theta/2} d\theta \ .$$

Consider the circle

$$C: \quad z = \beta_1 + \delta e^{i\theta}$$

with centre $(\beta_1,0)$ and radius δ . Since $\beta_1 + \delta < 1 < 1/\rho^2 < \beta_2$, C does not contain the points $(1/\rho^2,0)$ or $(\beta_2,0)$. Thus f(z) is analytic on and within C, and we may expand $f(\beta_1+\delta e^{i\theta})$ in a Taylor series about β_1 . It follows that

$$\begin{aligned} H &= \frac{i}{\sqrt{\delta}} \int_{-\pi}^{\pi} \left[\sum_{k=0}^{\infty} \frac{f^{(k)}(\beta_1)}{k!} (\delta e^{i\theta})^k \right] e^{-i\theta/2}\, d\theta \\ &= \frac{i}{\sqrt{\delta}} \sum_{k=0}^{\infty} \frac{f^{(k)}(\beta_1)}{k!} \delta^k \left[\frac{e^{i\theta(k-1/2)}}{i(k-1/2)} \right|_{-\pi}^{\pi} \\ &= \frac{1}{\sqrt{\delta}} \sum_{k=0}^{\infty} \frac{f^{(k)}(\beta_1)}{k!(k-1/2)} [e^{i\pi(k-1/2)} - e^{-i\pi(k-1/2)}] \ . \end{aligned}$$

Since $e^{i\pi(k-1/2)} = e^{-i\pi k} e^{-i\pi/2} = (-1)^k(-i)$ and $e^{-i\pi(k-1/2)} = e^{-i\pi k} e^{i\pi/2} = (-1)^k i$ we have

$$H = \frac{2i}{\sqrt{\delta}} \sum_{k=0}^{\infty} \frac{f^{(k)}(\beta_1)\delta^k}{k!(k-1/2)} (-1)^{k+1} .$$

Noting that for $k \geqslant 1$

$$\int_0^1 x^{k-3/2}\, dx = \left. \frac{x^{k-1/2}}{(k-1/2)} \right|_0^1 = \frac{1}{k-1/2} ,$$

we obtain

$$H = \frac{2i}{\sqrt{\delta}} \left[2f(\beta_1) + \sum_{k=1}^{\infty} \frac{f^{(k)}(\beta_1)\delta^k}{k!} (-1)^{k+1} \int_0^1 x^{k-3/2} dx \right]$$

$$= \frac{4i}{\sqrt{\delta}} f(\beta_1) + \frac{2i}{\sqrt{\delta}} \int_0^1 -x^{-3/2} \sum_{k=1}^{\infty} \frac{f^{(k)}(\beta_1)}{k!} (-x\delta)^k dx .$$

But the Taylor series expansion of $f(\beta_1 - x\delta)$ about β_1 is

$$f(\beta_1 - x\delta) = f(\beta_1) + \sum_{k=1}^{\infty} \frac{f^{(k)}(\beta_1)}{k!} (-x\delta)^k \, dx ,$$

and so, in the limit as $\varepsilon \to 0$,

$$\text{(3.8.9)} \quad \int_{\gamma^*} \frac{f(z)}{(z-\beta_1)^{3/2}} dz = \frac{4i}{\sqrt{\delta}} f(\beta_1) - \frac{2i}{\sqrt{\delta}} \int_0^1 x^{-3/2} [f(\beta_1 - x\delta) - f(\beta_1)] dx \cdot$$

Observing that as $\varepsilon \to 0$, $\Gamma^{*'} \to \Gamma^*$, and combining equations (3.8.6) through (3.8.9) we obtain

(3.8.10)
$$\int_{\Gamma^*} \frac{f(z)}{(z-\beta_1)^{3/2}}\, dz = 2i \int_\delta^{\beta_1} \frac{f(\beta_1-\tau)}{\tau^{3/2}}\, d\tau - \frac{4i}{\sqrt{\delta}}\ f(\beta_1)$$
$$+ \frac{2i}{\sqrt{\delta}} \int_0^1 x^{-3/2}[f(\beta_1-x\delta) - f(\beta_1)]dx\ .$$

Consider

$$H_1 = 2i \int_\delta^{\beta_1} \frac{f(\beta_1-\tau)}{\tau^{3/2}}\, d\tau\ .$$

Letting $y = \tau/\beta_1$, we have $\tau = \beta_1 y$, $d\tau = \beta_1 dy$, and

$$H_1 = \frac{2i}{\sqrt{\beta_1}} \int_{\delta/\beta_1}^1 y^{-3/2} f(\beta_1-\beta_1 y)\, dy$$
$$= \frac{2i}{\sqrt{\beta_1}} \int_{\delta/\beta_1}^1 y^{-3/2}[f(\beta_1-\beta_1 y)-f(\beta_1)]dy + \frac{2if(\beta_1)}{\sqrt{\beta_1}} \int_{\delta/\beta_1}^1 y^{-3/2} dy\ .$$

Now

$$\int_{\delta/\beta_1}^1 y^{-3/2}[f(\beta_1-\beta_1 y) - f(\beta_1)]dy$$
$$= \int_0^1 y^{-3/2}[f(\beta_1-\beta_1 y) - f(\beta_1)]dy - H_2\ ,$$

where

$$H_2 = \int_0^{\delta/\beta_1} y^{-3/2}[f(\beta_1-\beta_1 y) - f(\beta_1)]dy\ .$$

Substituting $x = \beta_1 y/\delta$ we see that $y = \delta x/\beta_1$, $dy = (\delta/\beta_1)dx$, and

$$H_2 = \int_0^1 \left(\frac{\delta x}{\beta_1}\right)^{-3/2}[f(\beta_1-\delta x) - f(\beta_1)]\,\frac{\delta}{\beta_1}\,dx$$

$$= \frac{\sqrt{\beta_1}}{\sqrt{\delta}} \int_0^1 x^{-3/2}[f(\beta_1-\delta x) - f(\beta_1)]\,dx \ .$$

Furthermore,

$$\int_{\delta/\beta_1}^1 y^{-3/2}dy = -2y^{-1/2}\Big|_{\delta/\beta_1}^1 = -2\left[1 - \frac{\sqrt{\beta_1}}{\sqrt{\delta}}\right] = 2\,\frac{\sqrt{\beta_1}}{\sqrt{\delta}} - 2 \ ,$$

and thus

$$H_1 = \frac{2i}{\sqrt{\beta_1}}\left[\int_0^1 y^{-3/2}[f(\beta_1-\beta_1 y) - f(\beta_1)]dy - \frac{\sqrt{\beta_1}}{\sqrt{\delta}}\int_0^1 x^{-3/2}[f(\beta_1-\delta x) - f(\beta_1)]dx\right] + \frac{2if(\beta_1)}{\sqrt{\beta_1}}\left[\frac{2\sqrt{\beta_1}}{\sqrt{\delta}} - 2\right].$$

From equation (3.8.10) it follows that

$$(3.8.11) \qquad \int_{\Gamma^*} \frac{f(z)}{(z-\beta_1)^{3/2}}\,dz = -\frac{4if(\beta_1)}{\sqrt{\beta_1}} + \frac{2i}{\sqrt{\beta_1}}\int_0^1 y^{-3/2}[f(\beta_1-\beta_1 y) - f(\beta_1)]dy \quad .$$

Consider

$$(3.8.12) \qquad I = \frac{1}{\sqrt{\beta_1}}\int_0^1 y^{-3/2}[f(\beta_1-\beta_1 y) - f(\beta_1)]dy \ .$$

From equation (3.8.5) we have

$$f(\beta_1-\beta_1 y) = f[\beta_1(1-y)]$$

$$= \beta_1^{(n-2)/2}(1-y)^{(n-2)/2} \frac{[1-\rho\beta_1(1-y)]^{5/2}[1+\rho\beta_1(1-y)]^{3/2}}{[1-\rho^2\beta_2(1-y)][\beta_1(1-y)-\beta_2]^{3/2}}$$

and

$$f(\beta_1) = \beta_1^{(n-2)/2} \frac{(1-\rho\beta_1)^{5/2}(1+\rho\beta_1)^{3/2}}{(1-\rho^2\beta_1)(\beta_1-\beta_2)^{3/2}} ,$$

so that

$$f(\beta_1-\beta_1 y) - f(\beta_1) = \beta_1^{(n-2)/2}[(1-y)^{(n-2)/2}g(y) - g(0)]$$

and

$$I = \int_0^1 \beta_1^{(n-3)/2} y^{-3/2}[(1-y)^{(n-2)/2}g(y) - g(0)]dy ,$$

where

$$g(y) = \frac{[1-\rho\beta_1(1-y)]^{5/2}[1+\rho\beta_1(1-y)]^{3/2}}{[1-\rho^2\beta_1(1-y)]\ [\beta_1(1-y)-\beta_2]^{3/2}} .$$

Expanding $g(y)$ in a Maclaurin series, which has radius of convergence > 1, we obtain

$$I = \beta_1^{(n-3)/2} \int_0^1 y^{-3/2}\left[(1-y)^{(n-2)/2} \sum_{k=0}^{\infty} \frac{g^{(k)}(0)y^{(k)}}{k!} - g(0)\right] dy$$

or

$$(3.8.13) \qquad I = \beta_1^{(n-3)/2}\left[g(0)I_0 + \sum_{k=1}^{\infty} \frac{g^{(k)}(0)}{k!} I_k\right]$$

where

$$I_0 = \int_0^1 y^{-3/2}[(1-y)^{(n-2)/2} - 1]dy$$

and

$$I_k = \int_0^1 y^{k-3/2}(1-y)^{(n-2)/2}dy \quad , \quad \text{for} \quad k = 1,2,\ldots .$$

Now for $k = 1,2,\ldots$

$$I_k = \int_0^1 y^{(k-1/2)-1}(1-y)^{n/2-1}dy ,$$

which is a beta function with parameters $(k-1/2)$ and $n/2$. Thus

$$(3.8.14) \qquad I_k = B(k-1/2,n/2) = \frac{\Gamma(k-1/2)\Gamma(n/2)}{\Gamma(k-1/2+n/2)} \quad , \quad \text{for} \quad k = 1,2,\ldots .$$

We may integrate by parts to evaluate I_0. Letting

$$u = (1-y)^{(n-2)/2} - 1 \quad \text{and} \quad dv = y^{-3/2}\, dy$$

we have

$$du = -\frac{(n-2)}{2}(1-y)^{(n-4)/2}dy \quad \text{and} \quad v = -2y^{-1/2}$$

so that

$$I_0 = \lim_{\varepsilon\to 0} \left. \frac{-2[(1-y)^{(n-2)/2}-1]}{y^{1/2}} \right|_{\varepsilon}^{1} - (n-2)\int_0^1 y^{-1/2}(1-y)^{(n-4)/2}dy \; .$$

Now

$$\lim_{\varepsilon\to 0} \left. \frac{-2[(1-y)^{(n-2)/2}-1]}{y^{1/2}} \right|_{\varepsilon}^{1} = 2 - \lim_{\varepsilon\to 0} \frac{-2[(1-\varepsilon)^{(n-2)/2} - 1]}{\varepsilon^{1/2}} \; ,$$

which upon application of L'Hopital's rule becomes

$$2 - \lim_{\varepsilon\to 0} \frac{(n-2)(1-\varepsilon)^{(n-4)/2}}{\varepsilon^{-1/2}/2} = 2 \; .$$

Furthermore

$$\int_0^1 y^{-1/2}(1-y)^{(n-4)/2}dy = \int_0^1 y^{1/2-1}(1-y)^{(n-2)/2-1}dy \; ,$$

which is a beta function with parameters 1/2 and (n-2)/2. Thus

$$\int_0^1 y^{-1/2}(1-y)^{(n-4)/2} = B[1/2,(n-2)/2] = \frac{\Gamma(1/2)\Gamma[(n-2)/2]}{\Gamma[1/2+(n-2)/2]} \; ,$$

and

$$I_0 = 2 - (n-2)\frac{\Gamma(1/2)\Gamma[(n-2)/2]}{\Gamma[(n-1)/2]} \; .$$

Recalling that $\Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1)$, it follows that

$$(3.8.15) \quad I_0 - 2 = -(n-2)\,\frac{\Gamma(1/2)\Gamma[(n-2)/2]}{\Gamma[(n-1)/2]}$$

$$= -\frac{2(n-2)}{2}\,\frac{\Gamma(1/2)\Gamma[(n-2)/2]}{\Gamma[(n-1)/2]}$$

$$= -2\,\frac{\Gamma(1/2)\Gamma(n/2)}{\Gamma[(n-1)/2]} \ .$$

Using equations (3.8.14) and (3.8.15), we obtain

$$I_1 = \frac{\Gamma(1/2)\Gamma(n/2)}{\Gamma[(n+1)/2]} = \frac{\Gamma(1/2)\Gamma(n/2)}{[(n-1)/2]\Gamma[(n-1)/2]} = \frac{-1}{(n-1)}\,(I_0-2) \ ,$$

$$I_2 = \frac{\Gamma(3/2)\Gamma(n/2)}{\Gamma[(n+3)/2]} = \frac{(1/2)\Gamma(1/2)\Gamma(n/2)}{[(n+1)/2]\Gamma[(n+1)/2]} = \frac{1}{(n+1)}\,I_1 = \frac{-1\cdot 1}{(n-1)(n+1)}\,(I_0-2)$$

$$I_3 = \frac{\Gamma(5/2)\Gamma(n/2)}{\Gamma[(n+5)/2]} = \frac{(3/2)\Gamma(3/2)\Gamma(n/2)}{[(n+3)/2]\Gamma[(n+3)/2]}$$

$$= \frac{3}{(n+3)}\,I_2 = \frac{-1\cdot 1\cdot 3}{(n-1)(n+1)(n+3)}\,(I_0-2)$$

and, in general, for $k = 1,2,\ldots$.

$$I_k = \frac{-1\cdot 1\cdot 3\cdots(2k-3)}{(n-1)(n+1)(n+3)\cdots(n+2k-3)}(I_0-2) \ ,$$

where $(I_o-2) = \dfrac{-2\Gamma(1/2)\Gamma(n/2)}{\Gamma[(n-1)/2]} = \dfrac{-2\sqrt{\pi}\ \Gamma(n/2)}{\Gamma[(n-1)/2]}$. From equation (3.8.13) it follows that

(3.8.16) $$I = \beta_1^{(n-3)/2}\{g(0)I_0$$

$$+ \sum_{k=1}^{\infty} \frac{g^{(k)}(0)}{k!} \frac{(-1)\cdot 1\cdot 3\ldots(2k-3)(I_0-2)}{(n-1)(n+1)(n+3)\ldots(n+2k-3)}\} \quad .$$

Combining equations (3.8.5), (3.8.11), (3.8.12), and (3.8.16) we obtain

$$h(r) \approx \frac{(1-\rho^2)^{1/2}}{4\pi\rho^3\sqrt{nr}}\left\{\frac{-4if(\beta_1)}{\sqrt{\beta_1}}\right.$$

$$\left.+ 2i\left[\beta_1^{(n-3)/2}\left(g(0)I_0 + \sum_{k=1}^{\infty} \frac{g^{(k)}(0)}{k!} \frac{(-1)\cdot 1\cdot 3\cdots(2k-3)(I_0-2)}{(n-1)(n+1)(n+3)\cdots(n+2k-3)}\right)\right]\right\}$$

$$= \frac{i(1-\rho^2)^{1/2}}{2\pi\rho^3\sqrt{nr}}\left\{-2\beta_1^{(n-3)/2}g(0)\right.$$

$$\left.+ \beta_1^{(n-3)/2}\left[g(0)I_0 + \sum_{k=1}^{\infty} \frac{g^{(k)}(0)}{k!} \frac{(-1)\cdot 1\cdot 3\cdots(2k-3)(I_0-2)}{(n-1)(n+1)(n-3)\cdots(n+2k-3)}\right]\right\}$$

$$= \frac{i(1-\rho^2)^{1/2}}{2\pi\rho^3\sqrt{nr}}\ \beta_1^{(n-3)/2}[I_0-2]\left\{g(0)\right.$$

$$\left.+ \sum_{k=1}^{\infty} \frac{g^{(k)}(0)}{k!} \frac{(-1)\cdot 1\cdot 3\cdots(2k-3)}{(n-1)(n+1)(n+3)\cdots(n+2k-3)}\right\} .$$

Since $g^{(k)}(0)$ depends on n only through β_1, which is bounded between 0 and 1, it follows that

$$\sum_{k=1}^{\infty} \frac{g^{(k)}(0)}{k!} \frac{(-1)\cdot 1\cdot 3\cdots(2k-3)}{(n-1)(n+1)(n+3)\cdots(n+2k-3)} = O(n^{-1}) \quad .$$

Furthermore, in approximating $|\underline{B}|$ we incurred errors which are relatively $O(n^{-1})$. Thus

$$h(r) \approx \frac{i(1-\rho^2)^{1/2}}{2\pi\rho^3\sqrt{nr}}\,\beta_1^{(n-3)/2}[I_0-2]g(0)\{1 + O(n^{-1})\}\ ,$$

where

$$g(0) = \frac{(1-\rho\beta_1)^{5/2}(1+\rho\beta_1)^{3/2}}{(1-\rho^2\beta_1)\quad(\beta_1-\beta_2)^{3/2}} = -\,\frac{(1-\rho\beta_1)^{5/2}(1+\rho\beta_1)^{3/2}}{i(1-\rho^2\beta_1)(\beta_2-\beta_1)^{3/2}}$$

and

$$[I_0-2] = -\,\frac{2\sqrt{\pi}\ \Gamma(n/2)}{\Gamma[(n-1)/2]}\ .$$

Noting that

$$(\beta_2-\beta_1)^{-3/2} = \left(\frac{1}{\rho^2\beta_1} - \beta_1\right)^{-3/2} = \left(\frac{1 - \rho^2\beta_1^2}{\rho^2\beta_1}\right)^{-3/2} = \frac{\rho^3\beta_1^{3/2}}{(1-\rho^2\beta_1^2)^{3/2}}\ ,$$

we see that

$$g(0) = -\,\frac{\rho^3\beta_1^{3/2}(1-\rho\beta_1)^{5/2}(1+\rho\beta_1)^{3/2}}{i(1-\rho^2\beta_1)(1-\rho^2\beta_1^2)^{3/2}} = -\,\frac{\rho^3\beta_1^{3/2}(1-\rho\beta_1)}{i(1-\rho^2\beta_1)}\ .$$

Therefore, for $0 < |\rho| < 1$, we have the following first approximation to the probability density of r :

$$\text{(3.8.17)}\qquad h(r) \approx \left[\frac{(1-\rho^2)^{1/2}}{\sqrt{n\pi}}\ \frac{\Gamma(n/2)}{\Gamma[(n-1)/2]}\right]\frac{\beta_1^{n/2}}{\sqrt{r}}\ \frac{(1-\rho\beta_1)}{(1-\rho^2\beta_1)}\ \{1 + O(n^{-1})\}\ ,$$

where

$$\beta_1 = \frac{n(1+\rho^2) + r(1-\rho)^2 - \sqrt{[n(1+\rho^2) + r(1-\rho)^2]^2 - 4n^2\rho^2}}{2n\rho^2} .$$

In this approximation for $h(r)$, each omitted term involves a factor of ρ. Also, by twice applying L'Hopital's rule it may be shown that

$$\lim_{\rho\to 0} \beta_1 = \frac{n}{(n+r)} .$$

It follows that

$$\lim_{\rho\to 0} h(r) = \frac{\Gamma(n/2)}{\sqrt{n\pi}\,\Gamma[(n-1)/2]} \frac{\left(\frac{n}{n+r}\right)^{n/2}}{\sqrt{r}} = \frac{\Gamma(n/2)}{\sqrt{n\pi r}\,\Gamma[(n-1)/2]} \frac{1}{(1+r/n)^{n/2}} ,$$

which, as shown in section 3.7, is the exact distribution of $h(r)$ in the special case $\rho = 0$.

3.9 A Renormalized Asymptotic Approximation for $h(r)$. The Approximate Moments of $h(r)$.

From the approximation (3.8.17) we see that for $0 < |\rho| < 1$

$$h(r) = K_n \frac{\beta_1^{n/2}}{\sqrt{r}} \frac{(1-\rho\beta_1)}{(1-\rho^2\beta_1)} \{1 + O(n^{-1})\} , \tag{3.9.1}$$

where K_n is a normalizing constant. The value of K_n may be determined by studying $\phi(\beta_1)$, the probability density of β_1 .

By equation (3.8.3) we have

$$\beta_1 = \frac{n(1+\rho^2) + r(1-\rho)^2 - \sqrt{[n(1+\rho^2) + r(1-\rho)^2]^2 - 4n^2\rho^2}}{2n\rho^2}$$

and

$$\beta_2 = \frac{n(1+\rho^2) + r(1-\rho)^2 + \sqrt{[n(1+\rho^2) + r(1-\rho)^2]^2 - 4n^2\rho^2}}{2n\rho^2} ,$$

so that

$$\beta_1 + \beta_2 = \frac{n(1+\rho^2) + r(1-\rho)^2}{n\rho^2} .$$

But since $\beta_2 = (\rho^2\beta_1)^{-1}$, we also have

$$\beta_1 + \beta_2 = \frac{\rho^2\beta_1 + 1}{\rho^2\beta_1} .$$

It follows that

$$r = \frac{n\rho^2(\beta_1+\beta_2) - n(1+\rho^2)}{(1-\rho)^2} = \left[n\rho^2\left(\frac{\rho^2\beta_1^2+1}{\rho^2\beta_1}\right) - n(1+\rho^2)\right] \Big/ (1-\rho)^2 ,$$

which, upon simplification, becomes

$$r = \frac{n(1-\beta_1)(1-\rho^2\beta_1)}{(1-\rho)^2\beta_1} . \tag{3.9.2}$$

Furthermore,

$$\frac{dr}{d\beta_1} = \frac{[-n(1-\rho^2\beta_1) + n(1-\beta_1)(-\rho^2)](1-\rho)^2\beta_1 - n(1-\beta_1)(1-\rho^2\beta_1)(1-\rho)^2}{(1-\rho)^4\beta_1^2} ,$$

which reduces to

$$\frac{dr}{d\beta_1} = \frac{-n(1-\rho B_1)(1+\rho B_1)}{(1-\rho)^2\beta_1^2} . \tag{3.9.3}$$

The probability density $\phi(\beta_1)$ is related to $h(r)$ by

$$\phi(\beta_1) = h(r)\left|\frac{dr}{d\beta_1}\right| = \frac{n(1-\rho\beta_1)(1+\rho\beta_1)}{(1-\rho)^2\beta_1^2}\, h\left[\frac{n(1-\beta_1)(1-\rho^2\beta_1)}{(1-\rho)^2\beta_1}\right], \tag{3.9.4}$$

for $0 < \beta_1 \leqslant 1$.

Combining equations (3.9.1) and (3.9.4) we obtain

$$\phi(\beta_1) = \frac{n(1-\rho\beta_1)(1+\rho\beta_1)}{(1-\rho)^2\beta_1^2} K_n\beta_1^{n/2}\left[\frac{n(1-\beta_1)(1-\rho^2\beta_1)}{(1-\rho)^2\beta_1}\right]^{-1/2} \frac{(1-\rho\beta_1)}{(1-\rho^2\beta_1)} \{1 + O(n^{-1})\}$$

$$= \frac{\sqrt{n}}{(1-\rho)} K_n \frac{(1-\rho\beta_1)^2(1+\rho\beta_1)\beta_1^{(n-3)/2}}{(1-\beta_1)^{1/2}(1-\rho^2\beta_1)^{3/2}} \{1 + O(n^{-1})\}$$

or

$$(3.9.5) \qquad \phi(\beta_1) = \frac{\sqrt{n}}{(1-\rho)} K_n \frac{(1-\rho\beta_1-\rho^2\beta_1^2+\rho^3\beta_1^3)\beta_1^{(n-3)/2}}{(1-\beta_1)^{1/2}(1-\rho^2\beta_1)^{3/2}} \{1 + O(n^{-1})\}$$

for $0 < \beta_1 \leqslant 1$.

Consider

$$J = \int_0^1 \frac{(1-\rho\beta_1-\rho^2\beta_1^2+\rho^3\beta_1^3)\beta_1^{(n-3)/2}}{(1-\beta_1)^{1/2}(1-\rho^2\beta_1)^{3/2}} d\beta_1 = J_1 - \rho J_2 - \rho^2 J_3 + \rho^3 J_4 ,$$

where

$$J_1 = \int_0^1 \frac{\beta_1^{(n-3)/2}}{(1-\beta_1)^{1/2}(1-\rho^2\beta_1)^{3/2}} d\beta_1 , \quad J_2 = \int_0^1 \frac{\beta_1^{(n-1)/2}}{(1-\beta_1)^{1/2}(1-\rho^2\beta_1)} d\beta_1$$

$$J_3 = \int_0^1 \frac{\beta_1^{(n+1)/2}}{(1-\beta_1)^{1/2}(1-\rho^2\beta_1)^{3/2}} d\beta_1 \quad , \text{ and}$$

$$J_4 = \int_0^1 \frac{\beta_1^{(n+3)/2}}{(1-\beta_1)^{1/2}(1-\rho^2\beta_1)^{3/2}} d\beta_1 .$$

From equation 3 of section 3.197 on page 286 of Gradshteyn and Ryzhik (1980), we have, for $Re(a) > 0$, $Re(b) > 0$, and $|d| < 1$, that

$$(3.9.6) \qquad \int_0^1 y^{a-1}(1-y)^{b-1}(1-dy)^{-c}dy = B(a,b)F(a,c;a+b;d)$$

where $B(\alpha,B)$ is the beta function and $F(\alpha,\beta;\gamma;z)$ is the

hypergeometric series

$$F(\alpha,\beta;\gamma;z) = 1 + \frac{\alpha\beta z}{\gamma\cdot 1} + \frac{\alpha(\alpha+1)\beta(\beta+1)z^2}{\gamma(\gamma+1)\cdot 1\cdot 2}$$

$$+ \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)z^3}{\gamma(\gamma+1)(\gamma+2)\cdot 1\cdot 2\cdot 3} + \dots \quad .$$

Thus

$$J_1 = B\left(\frac{n-1}{2}, \frac{1}{2}\right) F\left(\frac{3}{2}, \frac{n-1}{2}; \frac{n}{2}; \rho^2\right)$$

$$J_2 = B\left(\frac{n+1}{2}, \frac{1}{2}\right) F\left(\frac{3}{2}, \frac{n+1}{2}; \frac{n+2}{2}; \rho^2\right)$$

$$J_3 = B\left(\frac{n+3}{2}, \frac{1}{2}\right) F\left(\frac{3}{2}, \frac{n+3}{2}; \frac{n+4}{2}; \rho^2\right), \text{ and}$$

$$J_4 = \beta\left(\frac{n+5}{2}, \frac{1}{2}\right) F\left(\frac{3}{2}, \frac{n+5}{2}; \frac{n+6}{2}; \rho^2\right).$$

Furthermore

$$B\left(\frac{n-1}{2}, \frac{1}{2}\right) = \frac{\Gamma[(n-1)/2]\Gamma(1/2)}{\Gamma(n/2)} = \frac{\sqrt{\pi}\Gamma[(n-1)/2]}{\Gamma(n/2)} ,$$

$$B\left(\frac{n+1}{2}, \frac{1}{2}\right) = \frac{\Gamma[(n+1)/2]\Gamma(1/2)}{\Gamma[(n+2)/2]} = \frac{(n-1)}{n} \frac{\sqrt{\pi}\Gamma[(n-1)/2]}{\Gamma(n/2)},$$

$$B\left(\frac{n+3}{2}, \frac{1}{2}\right) = \frac{\Gamma[(n+3)/2]\Gamma(1/2)}{\Gamma[n+4)/2]} = \frac{(n-1)(n+1)}{n(n+2)} \frac{\sqrt{\pi}\Gamma[(n-1)/2]}{\Gamma(n/2)}$$

and

$$B\left(\frac{n+5}{2}, \frac{1}{2}\right) = \frac{\Gamma[(n+5)/2]\Gamma(1/2)}{\Gamma[(n+6)/2]}$$

$$= \frac{(n-1)(n+1)(n+3)}{n(n+2)(n+4)} \frac{\sqrt{\pi}\Gamma[(n-1)/2]}{\Gamma(n/2)} \quad .$$

From equation (3.9.5) it follows that

$$(3.9.7) \qquad K_n = \frac{(1-\rho)}{\sqrt{n}} J^{-1}$$

$$= \frac{(1-\rho)}{\sqrt{n}} \frac{\Gamma(n/2)}{\sqrt{\pi}\Gamma[(n-1)/2]} \left[F\left(\frac{3}{2}, \frac{n-1}{2}; \frac{n}{2}; \rho^2\right) \right.$$

$$- \frac{\rho(n-1)}{n} F\left(\frac{3}{2}, \frac{n+1}{2}; \frac{n+2}{2}; \rho^2\right)$$

$$- \rho^2 \frac{(n-1)(n+1)}{n(n+2)} F\left(\frac{3}{2}, \frac{n+3}{2}; \frac{n+4}{2}; \rho^2\right)$$

$$\left. + \rho^3 \frac{(n-1)(n+1)(n+3)}{n(n+2)(n+4)} F\left(\frac{3}{2}, \frac{n+5}{2}; \frac{n+6}{2}; \rho^2\right) \right]^{-1}$$

Note that for $\rho = 0$,

$$K_n = \frac{\Gamma(n/2)}{\sqrt{n\pi}\ \Gamma[(n-1)/2]}$$

which is the normalizing constant of the exact distribution of r in the special case $(\rho=0)$

Let

$$(3.9.8) \qquad h^*(r) = K_n \frac{\beta_1^{n/2}(1-\rho\beta_1)}{\sqrt{r}\ (1-\rho^2\beta_1)} \quad .$$

It follows from equation (3.9.1) that the moments of $h^*(r)$ are asymptotic approximations for the moments of $h(r)$. Thus

$$E(r) \approx \int_0^\infty rh^*(r)\,dr$$

$$= \int_0^1 rh^*(r)\left|\frac{dr}{d\beta_1}\right| d\beta_1 ,$$

which,by equations (3.9.2), (3.9.3), and (3.9.8) becomes

$$E(r) \approx K_n \int_0^1 \left[\frac{n(1-\beta_1)(1-\rho^2\beta_1)}{(1-\rho)^2\beta_1}\right]^{1/2} \beta_1^{n/2} \frac{(1-\rho\beta_1)}{(1-\rho^2\beta_1)} \frac{n(1-\rho\beta_1)(1+\rho\beta_1)}{(1-\rho)^2\beta_1^2} d\beta_1$$

$$= \frac{n^{3/2}K_n}{(1-\rho)^3}\int_0^1 \frac{\beta_1^{(n-5)/2}(1-\beta_1)^{1/2}(1-\rho\beta_1)^2(1+\rho\beta_1)}{(1-\rho^2\beta_1)^{1/2}} d\beta_1$$

$$= \frac{n^{3/2}K_n}{(1-\rho)^3}\int_0^1 \frac{\beta_1^{(n-5)/2}(1-\beta_1)^{1/2}(1-\rho\beta_1-\rho^2\beta_1^2+\rho^3\beta_1^3)}{(1-\rho^2\beta_1)^{1/2}} d\beta_1 .$$

By equation (3.9.6) we obtain

$$E(r) \approx \frac{n^{3/2}K_n}{(1-\rho)^3}\left[B\left(\frac{n-3}{2}, \frac{3}{2}\right)F\left(\frac{1}{2}, \frac{n-3}{2}; \frac{n}{2}; \rho^2\right)\right.$$

$$- \rho B\left(\frac{n-1}{2}, \frac{3}{2}\right)F\left(\frac{1}{2}, \frac{(n-1)}{2}; \frac{n+1}{2}; \rho^2\right)$$

$$- \rho^2 B\left(\frac{n+1}{2}, \frac{3}{2}\right)F\left(\frac{1}{2}, \frac{n+1}{2}; \frac{n+4}{2}; \rho^2\right)$$

$$+ \rho^3 B\left(\frac{n+3}{2}, \frac{3}{2}\right)F\left(\frac{1}{2}, \frac{n+3}{2}; \frac{n+6}{2}; \rho^2\right) .$$

In general we see that

$$E(r^j) \approx \int_0^\infty r^j h^*(r)\, dr$$

$$= \int_0^1 r^j h^*(r) \left|\frac{dr}{d\beta_1}\right| d\beta_1$$

$$= K_n \int_0^1 \left[\frac{n(1-\beta_1)(1-\rho^2\beta_1)}{(1-\rho)^2\beta_1}\right]^{(2j-1)/2} \beta_1^{n/2} \frac{(1-\rho\beta_1)}{(1-\rho^2\beta_1)} \frac{n(1-\rho\beta_1)(1+\rho\beta_1)}{(1-\rho)^2\beta_1^2}\, d\beta_1$$

$$= \frac{n^{(2j+1)/2}}{(1-\rho)^{2j+1}} \int_0^1 \beta_1^{(n-2j-3)/2} (1-\beta_1)^{(2j-1)/2} (1-\rho^2\beta_1)^{(2j-3)/2}$$

$$\cdot\ (1-\rho\beta_1-\rho^2\beta_1^2+\rho^3\beta_1^3)\, d\beta_1$$

and so, for j satisfying $n - 2j - 3 > 0$, we have

(3.9.9) $E(r^j)$

$$\approx \frac{n^{(2j+1)/2}K_n}{(1-\rho)^{2j+1}} \left[B\left(\frac{n-2j-1}{2}, \frac{2j+1}{2}\right) F\left(\frac{-2j+3}{2}, \frac{n-2j-1}{2}; \frac{n}{2}; \rho^2\right)\right.$$

$$- \rho B\left(\frac{n-2j+1}{2}, \frac{2j+1}{2}\right) F\left(\frac{-2j+3}{2}, \frac{n-2j+1}{2}; \frac{n+2}{2}; \rho^2\right)$$

$$- \rho^2 B\left(\frac{n-2j+3}{2}, \frac{2j+1}{2}\right) F\left(\frac{-2j+3}{2}, \frac{n-2j+3}{2}; \frac{n+4}{2}; \rho^2\right)$$

$$\left.+ \rho^3 B\left(\frac{n-2j+5}{2}, \frac{2j+1}{2}\right) F\left(\frac{-2j+3}{2}; \frac{n-2j+5}{2}; \frac{n+6}{2}; \rho^2\right)\right].$$

3.10 <u>The Approximate Probability Density and Moments of F</u>

By equation (3.5.1) we see that the statistic

$$F = \frac{(n-1)}{n} r$$

has probability density

$$g(F) = \frac{n}{(n-1)} h\left[\frac{nF}{(n-1)}\right].$$

From equation (3.9.1) it follows that

$$g(F) = \frac{nK_n}{(n-1)} \beta_1^{n/2}\left[\frac{nF}{(n-1)}\right]^{-1/2} \frac{(1-\rho\beta_1)}{(1-\rho^2\beta_1)} \{1 + O(n^{-1})\} ,$$

where K_n is given by equation (3.9.7) and

$$\beta_1 = \frac{n(1+\rho^2) + r(1-\rho)^2 - \sqrt{[n(1+\rho^2) + r(1-\rho)^2]^2 - 4n^2\rho^2}}{2n\rho^2}$$

$$= \frac{n(1+\rho^2) + nF(1-\rho)^2/(n-1) - \sqrt{[n(1+\rho^2) + nF(1-\rho)^2/(n-1)]^2 - 4n^2\rho^2}}{2n\rho^2}$$

Thus, a renormalized asymptotic approximation to the probability density of F is

(3.10.1) $$g(F) = \sqrt{\frac{n}{(n-1)}} \frac{K_n\beta_1^{n/2}(1-\rho\beta_1)}{\sqrt{F}\,(1-\rho^2\beta_1)} \{1 + O(n^{-1})\}$$

where

$$K_n = \frac{(1-\rho)\ \Gamma(n/2)}{\sqrt{n/\pi}\ \Gamma[(n-1)/2]}\Bigg[F\left(\frac{3}{2}\ ,\ \frac{n-1}{2}\ ;\ \frac{n}{2}\ ;\ \rho^2\right)$$

$$- \frac{\rho(n-1)}{n}\ F\left(\frac{3}{2}\ ,\ \frac{n+1}{2}\ ;\ \frac{n+2}{2}\ ;\ \rho^2\right)$$

$$- \frac{\rho^2(n-1)(n+1)}{n(n+2)}\ F\left(\frac{3}{2}\ ,\ \frac{n+3}{2}\ ;\ \frac{n+4}{2}\ ;\ \rho^2\right)$$

$$+ \frac{\rho^3(n-1)(n+1)(n+3)}{n(n+2)(n+4)}\ F\left(\frac{3}{2}\ ,\ \frac{n+5}{2}\ ;\ \frac{n+6}{2}\ ;\ \rho^2\right)\Bigg]^{-1}$$

and

$$\beta_1 = \frac{(1+\rho^2) + F(1-\rho)^2/(n-1) - \sqrt{[(1+\rho^2) + F(1-\rho)^2/(n-1)]^2 - 4\rho^2}}{2\rho^2}\ .$$

Since

$$E(F^j) = \left[\frac{(n-1)}{n}\right]^j E(r^j)$$

we obtain, from equation (3.9.9), that the j^{th} moment of F is given by

$$E(F^j) \approx \frac{n^{1/2}(n-1)^j K_n}{(1-\rho)^{2j+1}}\Bigg[B\left(\frac{n-2j-1}{2}\ ,\ \frac{2j+1}{2}\right) F\left(\frac{-2j+3}{2}\ ,\ \frac{n-2j-1}{2}\ ;\ \frac{n}{2};\ \rho^2\right)$$

$$- \rho B\left(\frac{n-2j+1}{2}\ ,\ \frac{2j+1}{2}\right) F\left(\frac{-2j+3}{2}\ ,\ \frac{n-2j+1}{2}\ ;\ \frac{n+2}{2}\ ;\ \rho^2\right)$$

$$- \rho^2 B\left(\frac{n-2j+3}{2}\ ,\ \frac{2j+1}{2}\right) F\left(\frac{-2j+3}{2}\ ,\ \frac{n-2j+3}{2}\ ;\ \frac{n+4}{2}\ ;\ \rho^2\right)$$

$$+ \rho^3 B\left(\frac{n-2j+5}{2}\ ,\ \frac{2j+1}{2}\right) F\left(\frac{-2j+3}{2}\ ,\ \frac{n-2j+5}{2}\ ;\ \frac{n+6}{2}\ ;\ \rho^2\right)\Bigg],$$

for j such that $n - 2j - 3 > 0$.

3.11 Areas for Further Research

By analogy to Daniels' (1956) work on the approximate distribution of serial correlation coefficients, it may be possible to refine approximation (3.10.1) so that the error is relatively $O(n^{-3/2})$. It may also be possible to bound the terms omitted in approximating $g(F)$. The expression for K_n could be put in a series form and studied further.

For selected values of n and ρ, numerical integration techniques could be used to obtain the approximate distribution function of F. This could then be compared with a computer-simulated empirical distribution, and with the limiting distribution of F.

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APPENDIX I

EVALUATION OF A KEY DETERMINANT

We wish to evaluate a determinant of the form

$$(\mathrm{I}.1) \qquad A_n = \begin{vmatrix} f & b & & & & & \\ b & a & b & & & (c) & \\ & b & a & b & & & \\ & & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot & \\ & (c) & & & b & a & b \\ & & & & & b & f \end{vmatrix}_n .$$

The following method of evaluating A_n is based on the work of Dixon (1944) and Patton (1961).

Replacing f by $[a+(f-a)]$ in (I.1), we obtain

$$A_n = \begin{vmatrix} a+(f-a) & b & & & & & \\ b & a & b & & & (c) & \\ & b & a & b & & & \\ & & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot & \\ & (c) & & & b & a & b \\ & & & & & b & a+(f-a) \end{vmatrix}_n$$

$$= (f-a) \begin{vmatrix} a & b & & & & & \\ b & a & b & & & (c) & \\ & b & a & b & & & \\ & & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot & \\ & (c) & & & b & a & b \\ & & & & & b & a+(f-a) \end{vmatrix}_{n-1}$$

$$+ \begin{vmatrix} a & b & & & & & \\ b & a & b & & & (c) & \\ & b & a & b & & & \\ & & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot & \\ & (c) & & & b & a & b \\ & & & & & b & a+(f-1) \end{vmatrix}_n$$

Similarly,

$$\begin{vmatrix} a & b & & & & & \\ b & a & b & & (c) & & \\ & b & a & b & & & \\ & & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot & \\ (c) & & & & b & a & b \\ & & & & & b & a+(f-a) \end{vmatrix}_n$$

$$= (f-a)\begin{vmatrix} a & b & & & & & \\ b & a & b & & (c) & & \\ & b & a & b & & & \\ & & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot & \\ (c) & & & & b & a & b \\ & & & & & b & a \end{vmatrix}_{n-1} + \begin{vmatrix} a & b & & & & & \\ b & a & b & & (c) & & \\ & b & a & b & & & \\ & & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot & \\ (c) & & & & b & a & b \\ & & & & & b & a \end{vmatrix}_n .$$

Thus,

$$A_n = B_n + 2(f-a)B_{n-1} + (f-a)^2 B_{n-2} , \tag{I.2}$$

where

$$B_n = \begin{vmatrix} a & b & & & & & \\ b & a & b & & (c) & & \\ & b & a & b & & & \\ & & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot & \\ (c) & & & & b & a & b \\ & & & & & b & a \end{vmatrix}_n .$$

Now,

$$B_n = \begin{vmatrix} 1 & 1 & 1 & \cdots & & & 1 \\ 0 & a & b & & & (c) & \\ 0 & b & a & b & & & \\ \vdots & & \ddots & \ddots & \ddots & & \\ \vdots & & & \ddots & \ddots & \ddots & \\ \vdots & (c) & & & b & a & b \\ 0 & & & & & b & a \end{vmatrix}_{n+1} = \begin{vmatrix} 1 & 1 & 1 & \cdots & & & 1 \\ -c & a-c & b-c & & & (0) & \\ -c & b-c & a-c & b-c & & & \\ \vdots & & \ddots & \ddots & \ddots & & \\ \vdots & & & \ddots & \ddots & \ddots & \\ \vdots & (0) & & & b-c & a-c & b-c \\ -c & & & & & b-c & a-c \end{vmatrix}_{n+1}$$

$$-cR_1+R_2,\ -cR_1+R_3, \ldots, -cR_1+R_{n+1}$$

Letting $d = a - c$ and $e = b - c$, we obtain

$$B_n = \begin{vmatrix} d & e & & & & \\ e & d & e & & (0) & \\ & e & d & e & & \\ & & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \ddots & \\ & (0) & & e & d & e \\ & & & & e & d \end{vmatrix}_n + \begin{vmatrix} 0 & 1 & 1 & \cdots & & & 1 \\ -c & d & e & & & (0) & \\ -c & e & d & e & & & \\ \vdots & & \ddots & \ddots & \ddots & & \\ \vdots & & & \ddots & \ddots & \ddots & \\ \vdots & (0) & & & e & d & e \\ -c & & & & & e & d \end{vmatrix}_{n+1} .$$

Hence

$$\text{(I.3)} \qquad B_n = C_n - cD_{n+1},$$

where

$$C_n = \begin{vmatrix} d & e & & & & \\ e & d & e & & (0) & \\ & e & d & e & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & \ddots \\ & (0) & & e & d & e \\ & & & & e & d \end{vmatrix}_n$$

and

$$D_n = \begin{vmatrix} 0 & 1 & 1 & \cdot & \cdot & \cdot & 1 \\ 1 & d & e & & & (0) & \\ 1 & e & d & e & & & \\ \cdot & & \cdot & \cdot & \cdot & & \\ \cdot & & & \cdot & \cdot & \cdot & \\ \cdot & (0) & & & e & d & e \\ 1 & & & & & e & d \end{vmatrix}_n .$$

Now

$$C_n = d \begin{vmatrix} d & e & & & & & \\ e & d & e & & & (0) & \\ & e & d & e & & & \\ & & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot & \\ & (0) & & & e & d & e \\ & & & & & e & d \end{vmatrix}_{n-1} - e \begin{vmatrix} e & & & & & & \\ e & d & e & & & (0) & \\ & e & d & e & & & \\ & & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot & \\ & (0) & & & e & d & e \\ & & & & & e & d \end{vmatrix}_{n-1}$$

$$= dC_{n-1} - e^2 C_{n-2} ,$$

or

$$\text{(I.4)} \qquad C_{n+2} - dC_{n+1} + e^2 C_n = 0 .$$

Letting ∇ be the forward difference operator of the calculus of finite differences, defined by

$$\nabla[f(j)] = f(j+1) ,$$

equation (I.4) may be expressed as

(I.5) $$(\nabla^2 - d\nabla + e^2)C_n = 0 .$$

The auxiliary equation of (I.5),

$$x^2 - dx + e^2 = 0$$

has roots

$$u = \frac{d+\sqrt{d^2-4e^2}}{2} \quad \text{and} \quad v = \frac{d-\sqrt{d^2-4e^2}}{2}$$

Since (I.5) is a second order homogeneous difference equation whose auxiliary equation has distinct roots, u and v, it follows that the general solution of (I.5) is

(I.6) $$C_n = K_1u^n + K_2v^n .$$

Using the initial conditions

$$C_1 = K_1u + K_2v = d , \quad C_2 = K_1u^2 + K_2v^2 = d^2 - e^2 ,$$

we obtain

$$K_1 = \frac{\begin{vmatrix} d & v \\ d^2-e^2 & v^2 \end{vmatrix}}{\begin{vmatrix} u & v \\ u^2 & v^2 \end{vmatrix}} = \frac{dv^2-vd^2+ve^2}{uv^2-u^2v} = \frac{dv-d^2+e^2}{u(v-u)}$$

and

$$K_2 = \frac{\begin{vmatrix} u & d \\ u^2 & d^2-e^2 \end{vmatrix}}{\begin{vmatrix} u & v \\ u^2 & v^2 \end{vmatrix}} = \frac{ud^2-ue^2-u^2d}{uv^2-u^2v} = \frac{d^2-e^2-ud}{v(v-u)}$$

Noting that

$$dv - d^2 + e^2 = -u^2$$

and

$$d^2 - e^2 - ud = v^2 ,$$

we have that

(I.7) $$K_1 = \frac{u}{(u-v)} \quad \text{and} \quad K_2 = \frac{-v}{(u-v)} .$$

Combining (I.6) and (I.7) we obtain

(I.8) $$C_n = \frac{u^{n+1}-v^{n+1}}{(u-v)} .$$

Now, expanding D_n by the last row, we obtain

$$D_n = (-1)^{n-1}\begin{vmatrix} 1 & 1 & 1 & \cdots & & & 1 \\ d & e & & & & (0) & \\ e & d & d & & & & \\ & \cdot & \cdot & \cdot & & & \\ & & \cdot & \cdot & \cdot & & \\ & (0) & & e & d & e & \\ & & & & e & d & e \end{vmatrix}_{n-1}$$

$$- e\begin{vmatrix} 0 & 1 & 1 & \cdots & & & & 1 \\ 1 & d & e & & & & (0) & \\ 1 & e & d & e & & & & \\ \cdot & & \cdot & \cdot & \cdot & & & \\ \cdot & & & \cdot & \cdot & \cdot & & \\ \cdot & & & & e & d & e & \\ 1 & (0) & & & & e & e & e \end{vmatrix}_{n-1}$$

$$+ d\begin{vmatrix} 0 & 1 & 1 & \cdots & & & 1 \\ 1 & d & e & & & (0) & \\ 1 & e & d & e & & & \\ \cdot & & \cdot & \cdot & \cdot & & \\ \cdot & & & \cdot & \cdot & \cdot & \\ \cdot & (0) & & & e & d & e \\ 1 & & & & & e & d \end{vmatrix}_{n-1} ,$$

or

$$\text{(I.9)} \qquad D_n = (-1)^{n-1}E_{n-1} - eF_{n-1} + dD_{n-1} ,$$

where

$$E_n = \begin{vmatrix} 1 & d & e & & & & \\ 1 & e & d & e & & (0) & \\ 1 & & e & d & e & & \\ \cdot & & & \cdot & \cdot & \cdot & \\ \cdot & & & & \cdot & \cdot & \cdot \\ \cdot & (0) & & & & e & d \\ 1 & & & & & & e \end{vmatrix}_n \quad \text{and} \quad F_n = \begin{vmatrix} 0 & 1 & 1 & \cdots & & & 1 \\ 1 & d & e & & & (0) & \\ 1 & e & d & e & & & \\ \cdot & & & \cdot & \cdot & \cdot & \\ \cdot & & & e & d & e & \\ \cdot & (0) & & & e & d & \\ 1 & & & & & e & e \end{vmatrix}_n .$$

Expanding F_n by the last column, we get

$$F_n = (-1)^{n-1}\begin{vmatrix} 1 & d & e & & & & \\ 1 & e & d & e & & (0) & \\ \cdot & & \cdot & \cdot & \cdot & & \\ \cdot & & & \cdot & \cdot & \cdot & \\ \cdot & & & & e & d & e \\ & (0) & & & & e & d \\ 1 & & & & & & e \end{vmatrix}_{n-1}$$

$$+\, e\begin{vmatrix} 0 & 1 & 1 & \cdot & \cdot & \cdot & 1 \\ 1 & d & e & & & (0) & \\ 1 & e & d & e & & & \\ \cdot & & \cdot & \cdot & \cdot & & \\ \cdot & & & \cdot & \cdot & \cdot & \\ \cdot & (0) & & & e & d & e \\ 1 & & & & & e & d \end{vmatrix}_{n-1}$$

or

(I.10) $$F_n = (-1)^{n-1}E_{n-1} + eD_{n-1} .$$

Combining equations (I.9) and (I.10), we obtain

$$D_n = (-1)^{n-1}E_{n-1} - e[(-1)^{n-2}E_{n-2}+eD_{n-2}] + dD_{n-1}$$

or

(I.11) $$D_n - dD_{n-1} + e^2D_{n-2} = (-1)^{n-1}(E_{n-1}+eE_{n-2}) .$$

Expanding E_n by the last row, we obtain

$$E_n = (-1)^{n-1}\begin{vmatrix} d & e & & & & \\ e & d & e & & (0) & \\ & e & d & e & & \\ & & \ddots & \ddots & \ddots & \\ & (0) & & e & d & e \\ & & & & e & d \end{vmatrix}_{n-1}$$

$$+\, e\begin{vmatrix} 1 & d & e & & & \\ 1 & e & d & e & (0) & \\ \cdot & & e & d & e & \\ \cdot & & & \ddots & \ddots & \ddots \\ \cdot & (0) & & & e & d \\ 1 & & & & & e \end{vmatrix}_{n-1}$$

or

$$E_n = (-1)^{n-1}C_{n-1} + eE_{n-1} \,. \tag{I.12}$$

From equations (I.8) and (I.12), we get

$$E_n - eE_{n-1} = (-1)^{n-1}\,\frac{u^n - v^n}{(u-v)} \quad . \tag{I.13}$$

Equation (I.13) may be expressed as a second order difference equation, the solution of which is

$$E_n = \frac{e^n}{2e+d} + (-1)^{n-1}\left[\frac{e(u^n-v^n) + u^{n+1} - v^{n+1}}{(u-v)\ (2e+d)}\right]. \tag{I.14}$$

From equation (I.11) it follows that

$$D_{n+2} - dD_{n+1} + e^2D_n = (-1)^{n-1}(E_{n+1}+eE_n) \ ,$$

or

$$(\text{I.15}) \qquad (\nabla^2 - d\nabla + e^2)D_n = (-1)^{n-1}(E_{n+1} + eE_n) \ .$$

From equation (I.14) it may be shown that

$$(\text{I.16}) \qquad E_{n+1} + eE_n = \frac{2e^{n+1} + (-1)^n(u^{n+1} + v^{n+1})}{2e+d} \ .$$

Combining equations (I.15) and (I.16), we obtain

$$(\text{I.17}) \qquad (\nabla^2 - d\nabla + e^2)D_n = \frac{2(-e)^{n+1} - (u^{n+1} + v^{n+1})}{2e+d} \ .$$

To obtain the complementary function of equation (I.17), we consider the homogeneous difference equation

$$(\nabla^2 - d\nabla + e^2)D_n = 0 \ .$$

The corresponding auxiliary equation,

$$x^2 - dx + e^2 = 0$$

has roots u and v. Thus

$$D_n = \lambda_1 u^n + \lambda_2 v^n$$

is the complementary function of the complete solution of equation (I.17). To obtain a particular solution, we assume a solution of the form

$$D_n = P(-e)^{n+1} + Qnu^n + Rnv^n \ . \tag{I.18}$$

Then

$$(\nabla^2 - d\nabla + e^2)D_n = P(-e)^{n+3} + Q(n+2)u^{n+2} + R(n+2)v^{n+2}$$

$$- d[P(-e)^{n+2} + Q(n+1)u^{n+1} + R(n+1)v^{n+1}]$$

$$+ e^2[P(-e)^{n+1} + Qnu^n + Rnv^n] \ ,$$

from which it may be shown that

$$(\nabla^2 - d\nabla + e^2)D_n = -P(2e+d)(-e)^{n+2} + Q(u-v)u^{n+1} - R(u-v)v^{n+1} \ . \tag{I.19}$$

Equating coefficients in equations (I.17) and (I.19), we obtain

$$P = \frac{2}{e(2e+d)^2} \ , \quad Q = \frac{-1}{(u-v)(2e+d)} \ , \quad R = \frac{1}{(u-v)(2e+d)} \ ,$$

and equation (I.18) becomes

$$D_n = \frac{2(-e)^{n+1}}{e(2e+d)^2} - \frac{nu^n}{(u-v)(2e+d)} + \frac{nv^n}{(u-v)(2e+d)}$$

$$= \frac{2(-1)^{n-1}e^n}{(2e+d)^2} - n\left[\frac{u^n-v^n}{(2e+d)(u-v)}\right].$$

Thus the complete solution of equation (I.17) is

$$\text{(I.20)} \qquad D_n = \lambda_1 u^n + \lambda_2 v^n + \frac{2(-1)^{n-1}e^n}{(2e+d)^2} - n\left[\frac{u^n-v^n}{(2e+d)(u-v)}\right],$$

where we may use the initial conditions to solve for λ_1 and λ_2. By definition

$$D_1 = |0| = 0 \quad \text{and} \quad D_2 = \begin{vmatrix} 0 & 1 \\ 1 & d \end{vmatrix} = -1 ,$$

and, from equation (I.20),

$$D_1 = \lambda_1 u + \lambda_2 v + \frac{2e}{(2e+d)^2} - \frac{(u-v)}{(2e+d)(u-v)} = \lambda_1 u + \lambda_2 v - \frac{(u+v)}{(2e+d)^2}$$

and

$$D_2 = \lambda_1 u^2 + \lambda_2 v^2 - \frac{2e^2}{(2e+d)^2} - \frac{2(u^2-v^2)}{(2e+d)(u-v)} = \lambda_1 u^2 + \lambda_2 v^2 - \frac{(u^2+v^2)}{(2e+d)^2} - 1 .$$

By inspection we have

$$\lambda_1 = \frac{1}{(2e+d)^2} \quad \text{and} \quad \lambda_2 = \frac{1}{(2e+d)^2} ,$$

so that equation (I.20) becomes

$$\text{(I.21)} \qquad D_n = \frac{u^n+v^n+2(-1)^{n-1}e^n}{(2e+d)^2} - \frac{n(u^n-v^n)}{(u-v)(2e+d)} .$$

Combining equations (I.3), (I.8), and (I.21) we obtain

$$B_n = \frac{u^{n+1}-v^{n+1}}{u-v} - c\left\{\frac{u^{n+1}+v^{n+1}+2(-1)^n e^{n+1}}{(2e+d)^2} - \frac{(n+1)(u^{n+1}-v^{n+1})}{(u-v)(2e+d)}\right\},$$

or

$$\text{(I.22)} \qquad \begin{aligned} B_n = {} & \frac{u^{n+1}}{(u-v)(2e+d)^2}\left\{(2e+d)^2 - c\left[(u-v)-(n+1)(2e+d)\right]\right\} \\ & \frac{-v^{n+1}}{(u-v)(2e+d)^2}\left\{(2e+d)^2 + c\left[(u-v)+(n+1)(2e+d)\right]\right\} \\ & + \frac{2c(-e)^{n+1}}{(2e+d)^2} . \end{aligned}$$

Substituting this value of B_n in equation (I.2) we find that

$$\begin{aligned} A_n = {} & \frac{u^{n+1}}{(u-v)(2e+d)^2}\left\{(2e+d)^2 - c[(u-v) - (n+1)(2e+d)]\right\} \\ & + \frac{2(f-a)u^n}{(u-v)(2e+d)^2}\left\{(2e+d)^2 - c[(u-v) - n(2e+d)]\right\} \\ & + \frac{(f-a)^2u^{n-1}}{(u-v)(2e+d)^2}\left\{(2e+d)^2 - c[(u-v) - (n-1)(2e+d)]\right\} \\ & - \frac{v^{n+1}}{(u-v)(2e+d)^2}\left\{(2e+d)^2 + c[(u-v) + (n+1)(2e+d)]\right\} \end{aligned}$$

$$- \frac{2(f-a)v^n}{(u-v)(2e+d)^2}\left\{(2e+d)^2 + c[(u-v) + n(2e+d)]\right\}$$

$$- \frac{(f-a)^2 v^{n-1}}{(u-v)(2e+d)^2}\left\{(2e+d)^2 + c[(u-v) + (n-1)(2e+d)]\right\}$$

$$+ \frac{2c(-e)^{n+1}}{(2e+d)^2} + \frac{4(f-a)c(-e)^n}{(2e+d)^2} + \frac{2(f-a)^2 c(-e)^{n-1}}{(2e+d)^2} .$$

Collecting terms and simplifying we obtain

$$\text{(I.23)} \qquad A_n = \frac{u^{n-1}}{(u-v)(2e+d)^2}\left\{[u+(f-a)]^2[(2e+d)^2 - c\{(u-v)-n(2e+d)\}]\right.$$

$$\left. + c(2e+d)[u^2-(f-a)^2]\right\}$$

$$- \frac{v^{n-1}}{(u-v)(2e+d)^2}\left\{[v+(f-a)]^2[(2e+d)^2 + c\{(u-v)+n(2e+d)\}]\right.$$

$$\left. + c(2e+d)[v^2-(f-a)^2]\right\}$$

$$+ \frac{2c(-e)^{n-1}}{(2e+d)^2}[e-(f-a)]^2$$

where u and v are roots of the equation

$$x^2 - dx + e^2 = 0$$

and

$$d = a - c \,, \; e = b - c \,.$$

It is easily shown that A_n is symmetric in u and v.

APPENDIX II

THE JOUKOWSKI TRANSFORMATION

Consider the complex-valued mapping

(II.1) $$w = z + \frac{1}{z} ,$$

which is known as the Joukowski transformation. Letting $z = re^{i\theta}$ we have that

$$w = re^{-i\theta} + r^{-1}e^{-i\theta} = r(\cos\theta + i\sin\theta) + r^{-1}(\cos(-\theta) + i\sin(-\theta))$$

or

$$w = (r+r^{-1})\cos\theta + i(r-r^{-1})\sin\theta ,$$

which is a more convenient expression to study.

If $r = 1$ then $w = 2\cos\theta$. Thus, as z proceeds counterclockwise along the unit circle from (1,0) to (-1,0) to (1,0), w moves along the real axis in the w-plane from 2 to -2 and back to 2.

If z moves on a circle of radius $r \neq 1$ centred at the origin, then the locus of $w = u + iv$ is the ellipse

$$\frac{u^2}{a^2} + \frac{v^2}{b^2} = 1 ,$$

with $a = r + r^{-1}$ and $b = r - r^{-1}$. It follows that if $|z| = r > 1$, then: (i) as $r \to +\infty$, a and $b \to +\infty$; (ii) as $r \to 1$, $a \to 2$ and $b \to 0$; (iii) $a > 2$ and $b > 0$; (iv) $r > r^{-1}$; (v) for $0 < \theta < \pi$, w lies in the upper half of the w-plane; and (vi) for $\pi < \theta < 2\pi$, w lies in the lower half-plane. Thus the Joukowski transformation maps the w-plane, cut along the real axis from -2 to 2, inclusive, onto the exterior of the unit circle $|z| = 1$. Furthermore,

the upper half of the w-plane is mapped onto the upper half of the exterior of $|z| = 1$, and the lower half onto the lower half exterior.

Similarly, if $|z| = r < 1$, then: (i) as $r \to 0$, $a \to +\infty$ and $b \to -\infty$; (ii) as $r \to 1$, $a \to 2$ and $b \to 0$; (iii) $a > 2$ and $b < 0$; (iv) $r < r^{-1}$; (v) for $0 < \theta < \pi$, w lies in the lower half of the w-plane; and (vi) for $\pi < 0 < 2\pi$, w lies in the upper half-plane. Thus, the Joukowski transformation also maps the cut w-plane onto the interior of the unit circle $|z| = 1$, with the upper half of the w-plane going to the lower half interior and the lower half of the w-plane mapping to the upper half interior.

We may also express equation (II.1) as the quadratic equation

$$z^2 - wz + 1 = 0$$

with roots

$$z_1 = \frac{w - \sqrt{w^2-4}}{2} \text{ and } z_2 = \frac{w + \sqrt{w^2-4}}{2} .$$

Observing that $z_1 z_2 = 1$ we obtain

$$|z_1||z_2| = 1.$$

It follows that for given w either one solution of equation (II.1) lies inside the unit circle and the other outside, or both solutions lie on the unit circle.